Reachable subspaces, control regions and heat equations with memory

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Abstract

We study the controlled heat equations with analytic memory from two perspectives: reachable subspaces and control regions. Due to the hybrid parabolic-hyperbolic phenomenon of the equations, the support of a control needs to move in time to efficiently control the dynamics. We show that under a sharp sufficient geometric condition imposed to the control regions, the difference between reachable subspaces of the controlled heat equations, with and without memory, is precisely given by a Sobolev space. The appearance of this Sobolev space is attributed to the memory which makes the equation having the wave-like nature. The main ingredients in the proofs of our main results are as: first, the decomposition of the flow (generated by the equation with the null control) given in [31], second, an observability inequality built up in this paper.

Keywords. Controlled heat equations with memory, reachable subspaces, control regions

1 Introduction

We consider the following controlled heat equation with memory:

$$\begin{cases} \partial_t y(t,x) - \Delta y(t,x) + \displaystyle\int_0^t M(t-s)y(s,x)ds = \chi_Q(t,x)u(t,x), & (t,x)\in\mathbb{R}^+\times\Omega, \\ y(t,x)=0, & (t,x)\in\mathbb{R}^+\times\partial\Omega, \\ y(0,x)=y_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here, $\mathbb{R}^+ := (0,+\infty)$, $\Omega\subset\mathbb{R}^n$ ($n\in\mathbb{N}^+ := \{1,2,3,\cdots\}$) is a bounded domain with a C^2-boundary $\partial\Omega$, Q is a control region, which is a nonempty open subset in $\overline{\mathbb{R}^+}\times\overline{\Omega}$ (where $\overline{\mathbb{R}^+} := [0,+\infty)$ and $\overline{\Omega}$ is the closure of Ω in $\mathbb{R}^n$), χ_Q is the characteristic function of Q, y_0 is an initial datum, u is a control and M is a memory kernel. Throughout this paper, we assume

($\mathfrak{H}$) The memory kernel M is a real analytic and nonzero function over $\overline{\mathbb{R}^+}$.

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Mathematics Subject Classification (2010): 93B03 45K05 93B05 93B07 35K05

The aim of the paper is to study the equation (1.1) from two perspectives: reachable subspaces and control regions. To state our main results, we need the following definitions:

(D1) (*Reachable sets*) Given $p \in [1, +\infty]$, $S > 0$ and $y_0 \in L^2(\Omega)$, the following set is called a reachable set for the equation (1.1) at time S:

$$\mathcal{R}_M^p(S, y_0) := \big\{ y(S; y_0, u) \ : \ u \in L^p(\mathbb{R}^+; L^2(\Omega)) \big\}, \tag{1.2}$$

where $y(\cdot; y_0, u)$ is the solution of the equation (1.1), while the following set is called a reachable set for the equation (1.1) with $M = 0$ at time S:

$$\mathcal{R}_0^p(S, y_0) := \big\{ z(S; y_0, u) \ : \ u \in L^p(\mathbb{R}^+; L^2(\Omega)) \big\}, \tag{1.3}$$

where $z(\cdot; y_0, u)$ is the solution of the equation (1.1) with $M = 0$.

(D2) (*Moving control condition*, MCC for short) Let $T > 0$ and let Q be a nonempty open subset in $\overline{\mathbb{R}^+} \times \overline{\Omega}$. The pair (Q, T) is said to satisfy MCC, if for each $x \in \overline{\Omega}$, there is $t \in [0, T]$ so that $(t, x) \in Q$.

It deserves mentioning that in the above, $\overline{\mathbb{R}^+} \times \overline{\Omega}$ is viewed as a topology space with the topology induced from $\mathbb{R} \times \mathbb{R}^n$. Thus, the points in Q are allowed to be at the boundary of $\overline{\mathbb{R}^+} \times \overline{\Omega}$. This makes the description of MCC simpler.

(D3) (*Space $\mathcal{H}^s$*) Let η_j be the j^{th} eigenvalue of $-A$ and let e_j be the corresponding normalized eigenfunction. For each $s \in \mathbb{R}$, we define the real Hilbert space:

$$\mathcal{H}^s := \Big\{ f = \sum_{j=1}^{\infty} a_j e_j \ : \ (a_j)_{j \geq 1} \subset \mathbb{R}, \ \sum_{j=1}^{\infty} |a_j|^2 \eta_j^s < +\infty \Big\}, \tag{1.4}$$

equipped with the inner product:

$$\langle f, g \rangle_{\mathcal{H}^s} := \sum_{j=1}^{\infty} a_j b_j \eta_j^s, \qquad f = \sum_{j=1}^{\infty} a_j e_j \in \mathcal{H}^s \ \text{ and } \ g = \sum_{j=1}^{\infty} b_j e_j \in \mathcal{H}^s, \tag{1.5}$$

where $(a_j)_{j \geq 1}, (b_j)_{j \geq 1} \subset \mathbb{R}$.

Two main results of this paper are given in order.

Theorem 1.1. *Assume that (Q, T) satisfies MCC. Then for any $1 \leq p \leq +\infty$, $S \geq T$ and $y_0 \in L^2(\Omega)$,*

$$\mathcal{R}_M^p(S, y_0) = \mathcal{R}_0^p(S, y_0) + \mathcal{H}^4. \tag{1.6}$$

Theorem 1.2. *Assume that there is $p \in [1, +\infty]$ and $T > 0$ so that for any $y_0 \in L^2(\Omega)$,*

$$0 \in \mathcal{R}_M^p(T, y_0). \tag{1.7}$$

Then $(\overline{Q}, T)$ satisfies MCC, i.e., given $x \in \overline{\Omega}$, there is $t \in [0, T]$ so that $(t, x) \in \overline{Q}$. (Here $\overline{Q}$ is the closure of Q in $\mathbb{R} \times \mathbb{R}^n$.)

The next result is a direct consequence of Theorem 1.1.

Corollary 1.3. *Assume that (Q,T) satisfies MCC. Then given $S \geq T$, $y_0 \in L^2(\Omega)$ and $y_1 \in \mathcal{H}^4$, there is $u \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ so that $y(S; y_0, u) = y_1$.*

Several notes on the above results, as well as definitions, are as follows:

(a1) Theorem 1.1 can be understood in the following manner: The exact difference between reachable sets of the controlled heat equations with and without memory is the space $\mathcal{H}^4$, due to the wave-like effect of the system (1.1). This result seems to be new for us.

(a2) By Theorem 1.1, we can show that MCC for (Q,T) implies that each $\mathcal{R}_M^p(S, y_0)$, with $S \geq T$, is a linear subspace of $L^2(\Omega)$. (This will be shown in (iii) of Proposition 5.1 in Appendix 5.1.)

(a3) MCC for (Q,T) is a sharp sufficient condition on (1.6) (or (1.7)) in the following sense:

$$\text{MCC for } (Q,T) \Rightarrow (1.6) \text{ (or (1.7))}; \quad (1.6) \text{ (or (1.7))} \Rightarrow \text{MCC for } (\overline{Q},T).$$

(The above will be shown in (iv) and (v) of Proposition 5.1 in Appendix 5.1.)

(a4) Corollary 1.3 gives a kind of one-state exact controllability for the equation (1.1), while (1.7) means the one-state null controllability for the equation (1.1). Here, by the one-state controllability, we mean the usual controllability. We use this terminology to distinguish it from the memory-type controllability studied in [4, Definition 1.1].

(a5) The real analyticity of M is mainly used to prove some unique continuation property for the system (1.1) (see Lemma 3.4). It is open for us whether such unique continuation property holds for general memory kernels. It deserves mentioning the work [4] where for (1.1) with some polynomial-like memory kernel, the authors built up a Carleman estimate, from which the unique continuation follows at once. However, we do not know how to get the similar Carleman estimate for (1.1) with general analytic kernel M, and thus we show the unique continuation in another way.

(a6) The reachable sets studied in this paper do not correspond to controlling the full dynamics, which needs dealing with the memory terms as what have been done in [4] for (1.1) with polynomial-like kernel M. (See also [21] for the wave equation with memory.) It seems to be a challenging issue to study reachable sets with controlling the full dynamics. We are not clear how to define appropriately reachable sets with controlling the full dynamics for (1.1) where M is an analytic function.

(a7) Our MCC is originally from [4, Assumption 4.1], though it is strictly weaker than the later (see Example 5.2 in Appendix 5.2). But on the other hand, the sharpness of MCC for (Q,T) seems to be new. The proposal of MCC is due to the wave-like nature (for the equation (1.1)) which is manifested in the propagation of singularities along the time direction for the equation (1.1) (see [31]). Essentially, MCC for (Q, T) requires that each characteristic line must enter the control region Q before the time T. This is similar to (but looks simpler than) the well-known GCC for the classical wave equations (see, for instance, [27] and [1]).

We next introduce some related works. We begin with some works on the reachable subspaces for heat equations. For some one-dimensional boundary control heat equations, the reachable subspaces are shown to be time-invariant in [8, 23], and to include a subspace of some analytic functions in [9, 6]. Furthermore, it was proved in [22, 5, 16] that a reachable subspace is sandwiched between two spaces of analytic functions and the difference of these two spaces is *very small*. Recently, these results are extended to some high-dimensional boundary control heat equation over a ball in [28].

We then refer to some works on differential equations with persistent memory. (The equation (1.1) is one of them.) Such systems have been studied in many literatures (see, for instance, [2, 7, 14, 25, 29, 32] and their references therein). One of several intense topics about these equations is the controllability (see [25] for an overview).

We now introduce some works on the one-state controllability of differential equations with memory. The controllability and unique continuation for parabolic equations have been widely studied in literatures (see for instance, [10, 12, 18, 19, 26, 34]). However, the controllability for parabolic equations with memory is far from clear. Here we would like to mention what follows: The lack of the null controllability of such equations was studied in [13, 15, 33] and the references therein; The approximate controllability for the equation (1.1) (where $Q = \mathbb{R}^+ \times \omega$, with ω an open subset of Ω) was built up in [33]; The exact controllability of some parabolic-like equation with memory in the leading operator, was obtained in [11] where the control time needs to be larger and control region needs to satisfy some geometric condition; The spectral controllability was studied in [24] for a class of control systems with memory in a finite dimension setting.

Finally, we mention the memory-type null controllability. By our understanding, when built up the heat equation, people agreed that zero is a steady state. Later, people thought of that the heat equation cannot accurately simulate the thermal process and the heat equation with memory can do better. It is swing and roundabouts that people improved the model but lost the steady state zero. Thus, for the null controllability of the equation (1.1) with memory, it is not enough to consider only one state variable. In [4], the null controllability of two state variables (called as the memory-type null controllability) was studied for the equation (1.1) where M is a polynomial-like function. For the works in this direction, we also quote [3, 21] and the references therein.

The rest of this paper is organized as follows: Section 2 gives some preliminaries; Section 3 builds up an observability inequality; Section 4 proves the main theorems; Section 5 is the appendix.

2 Preliminaries

We recall [31, (1.2)] for the definition of the flow $\Phi(t)$: for each $t \geq 0$,

$$\Phi(t)y_0 := y(t; y_0, 0), \quad y_0 \in L^2(\Omega). \tag{2.1}$$

It follows by [31, Theorem 1.1] that each $\Phi(t)$ (with $t \geq 0$) can be treated as an element in the space $\mathcal{L}(H^s)$ for any $s \in \mathbb{R}$. Let

$$Af := \Delta f, \ \text{ with its domain } \ D(A) := H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

Write $\{e^{tA}\}_{t \geq 0}$ is the C_0 semigroup generated by the operator A.

In this section, we will first review some properties of the flow $\Phi(t)$, including a decomposition theorem for the flow, which are obtained in [31], and then give some consequences of the decomposition theorem.

2.1 Review on the decomposition of the flow

We start with recalling the following functions from [31]: (i) for each $l \in \mathbb{N}$, let

$$\begin{cases} h_l(t) := & (-1)^l \sum_{j=0}^{l} C_l^{l-j} \dfrac{d^{(l-j)}}{dt^{(l-j)}} \underbrace{M * \cdots * M}_{j}(t), \ t \geq 0, \\ p_l(t) := & -h_l(0) + (-1)^{l+1} \displaystyle\sum_{\substack{m,\, j \,\in\, \mathbb{N}^+, \\ 2j - l - 1 \leq m \leq j}} \left(C_l^{l-j+m} \dfrac{d^{(l-j+m)}}{dt^{(l-j+m)}} \underbrace{M * \cdots * M}_{j}(0) \right) \dfrac{(-t)^m}{m!}, \ t \geq 0, \end{cases} \tag{2.3}$$

where $C_\beta^m := \frac{\beta!}{m!(\beta-m)!}$ and $\underbrace{M * \cdots * M}_{j} := 0$ if $j = 0$; (ii) the flow kernel is defined by

$$K_M(t,s) := \sum_{j=1}^{+\infty} \frac{(-s)^j}{j!} \underbrace{M * \cdots * M}_{j}(t-s), \quad t \geq s. \tag{2.4}$$

By the assumption $(\mathfrak{H})$, the next decomposition theorem (i.e., Theorem 2.1), Proposition 2.2 and Proposition 2.3 follow from [31, Theorem 1.1], [31, Proposition 4.6] and [31, Proposition 4.7], respectively.

Theorem 2.1. *For each $N \in \mathbb{N}^+ \setminus \{1\}$, it holds that*

$$\Phi(t) = \mathfrak{P}_N(t) + \mathfrak{H}_N(t) + \mathfrak{R}_N(t), \quad t \geq 0, \tag{2.5}$$

with

$$\begin{cases} \mathfrak{P}_N(t) & := & e^{tA} + e^{tA} \sum_{l=0}^{N-1} p_l(t)(-A)^{-l-1}, \\ \mathfrak{H}_N(t) & := & \sum_{l=0}^{N-1} h_l(t)(-A)^{-l-1}, \\ \mathfrak{R}_N(t) & := & \mathcal{R}_N(t, -A)(-A)^{-N-1}, \end{cases} \qquad t \geq 0, \tag{2.6}$$

where

$$\mathcal{R}_N(t,\tau) := \int_0^t \tau e^{-\tau s}\partial_s^N K_M(t,s)ds, \quad t\geq 0, \quad \tau\geq 0.$$

Moreover, for each $s\in\mathbb{R}$, $\Phi(t)$, with $t\geq 0$, can be treated as an element in $\mathcal{L}(\mathcal{H}^s)$; and $\mathcal{R}_N(\cdot,-A)|_{\mathbb{R}^+}$ belongs to $C(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^s))$ and satisfies

$$\left\|\mathcal{R}_N(t,-A)\right\|_{\mathcal{L}(\mathcal{H}^s)} \leq e^t\left\{\exp\left[N(1+t)\left(\sum_{j=0}^N \max_{0\leq s\leq t}\left|\frac{d^j}{ds^j}M(s)\right|\right)\right]-1\right\}, \quad t\geq 0. \tag{2.7}$$

Besides, for each $t\geq 0$, $\{h_l(t)\}_{l\geq 1}$ is not the zero sequence.

Proposition 2.2. *Let K_M be given by (2.4). Then*

$$\Phi(t)^* = \Phi(t) = e^{tA} + \int_0^t K_M(t,\tau)e^{\tau A}d\tau, \ \ t\geq 0. \tag{2.8}$$

Proposition 2.3. *For each $s\in\mathbb{R}$, $\Phi(\cdot)$ is real analytic from $\mathbb{R}^+$ to $\mathcal{L}(\mathcal{H}^s)$.*

2.2 Some consequence of the decomposition theorem

This subsection gives two consequences of Theorem 2.1 which concerns the leading terms of the first two components in the decomposition (2.5) and will be used in the proof of our main theorems.

Corollary 2.4. *There is $\mathcal{R}_c\in C\big(\mathbb{R}^+;C(\mathbb{R}^+)\big)$ so that*

$$\Phi(t) = -M(t)A^{-2} + \mathcal{R}_c(t,-A)A^{-3}, \ \ t>0. \tag{2.9}$$

Moreover, for each $s\in\mathbb{R}$, $\mathcal{R}_c(\cdot,-A)$ belongs to $C(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^s))$ and satisfies

$$\left\|\mathcal{R}_c(t,-A)\right\|_{\mathcal{L}(\mathcal{H}^s)} \leq t^{-3}\exp\Big(2(1+t)\big(1+\|M\|_{C^2([0,t])}\big)\Big), \quad t>0. \tag{2.10}$$

Proof. By applying Theorem 2.1 whit $N=2$, we see that for each $t\geq 0$,

$$\begin{aligned}\Phi(t) = e^{tA} &+ e^{tA}\Big(-p_0(t)A^{-1}+p_1(t)A^{-2}\Big)\\ &+\Big(-h_0(t)A^{-1}+h_1(t)A^{-2}\Big) - \mathcal{R}_2(t,-A)A^{-3}.\end{aligned} \tag{2.11}$$

At the same time, it follows from (2.3) that for each $t\geq 0$,

$$\begin{cases} p_0(t) = M(0)t, \\ p_1(t) = M(0) - M'(0)t + \frac{1}{2}M(0)^2t^2, \\ h_0(t) = 0, \quad h_1(t) = -M(t). \end{cases} \tag{2.12}$$

This, along with (2.11), yields

$$\begin{aligned}\Phi(t) = &-M(t)A^{-2} - \mathcal{R}_2(t,-A)A^{-3}\\ &+e^{tA}\left[Id - M(0)tA^{-1} + \Big(M(0)-M'(0)t+\frac{1}{2}M(0)^2t^2\Big)A^{-2}\right], \ t\geq 0.\end{aligned} \tag{2.13}$$

Next, we define that for each $t > 0$,

$$\mathcal{R}_c(t, -A) := e^{tA}\Big[A^3 - M(0)tA^2 + \Big(M(0) - M'(0)t + \frac{1}{2}M(0)^2t^2\Big)A\Big] - \mathcal{R}_2(t, -A). \quad (2.14)$$

At the same time, one can directly check that for each $j \in \{1, 2, 3\}$ and each $s \in \mathbb{R}$,

$$\|A^j e^{tA}\|_{\mathcal{L}(\mathcal{H}^s)} \leq j^j e^{-j} t^{-j} \leq 2t^{-j}, \quad t > 0. \quad (2.15)$$

Now, (2.9) follows from (2.13) and (2.14), while (2.10) follows by (2.14), (2.15) and (2.7) with $N = 2$. This ends the proof of Corollary 2.4. □

Corollary 2.5. *There is $\widetilde{\mathcal{R}}_c \in C\big(\mathbb{R}^+; C(\mathbb{R}^+)\big)$ so that*

$$\Phi(t) = e^{tA} + \widetilde{\mathcal{R}}_c(t, -A)A^{-2}, \quad t \geq 0. \quad (2.16)$$

Moreover, for each $s \in \mathbb{R}$, $\widetilde{\mathcal{R}}_c(\cdot, -A)$ belongs to $C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^s))$ and satisfies that for some $C_1 > 0$,

$$\|\widetilde{\mathcal{R}}_c(t, -A)\|_{\mathcal{L}(\mathcal{H}^s)} \leq C_1 \exp\Big(2(1+t)\big(1 + \|M\|_{C^2([0,t])}\big)\Big), \quad t > 0.$$

Proof. From (2.11), it follows

$$\Phi(t) = e^{tA} + \widetilde{\mathcal{R}}_c(t, -A)A^{-2}, \ \ t \geq 0, \quad (2.17)$$

where

$$\widetilde{\mathcal{R}}_c(t, -A) := \big(p_1(t) - p_0(t)A\big) + e^{tA}\big(h_1(t) - h_0(t)A\big) - \mathcal{R}_2(t, -A)A^{-1}, \quad t \geq 0.$$

Then by (2.17), we get (2.16), with the above $\widetilde{\mathcal{R}}_c$.

Next, one can directly check that for each $s \in \mathbb{R}$, $\|tAe^{tA}\|_{\mathcal{L}(\mathcal{H}^s)} \leq 1$, when $t \geq 0$, which, along with (2.12) and (2.7), yields the desired estimate about the above $\widetilde{\mathcal{R}}_c$. This completes the proof of Corollary 2.5. □

2.3 Variation of constant formula

The aim of this subsection is to give the variation of constant formula for the equation (1.1) in the next Proposition 2.6. *It deserves mentioning what follows: this formula is not as easy as we think, since $\{\Phi(t)\}_{t\geq 0}$ has no the semigroup property: $\Phi(t+s) = \Phi(t)\Phi(s)$.*

Proposition 2.6. *When $y_0 \in L^2(\Omega)$ and $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$,*

$$y(t; y_0, u) = \Phi(t)y_0 + \int_0^t \Phi(t-s)(\chi_Q u)(s)ds, \ \ t \geq 0. \quad (2.18)$$

Proof. Arbitrarily fix $y_0 \in L^2(\Omega)$ and $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$. Simply write $y(\cdot)$ for the solution $y(\cdot; y_0, u)$. First of all, (2.18) is clearly true for $t = 0$. Now we arbitrarily fix $t > 0$ and $z \in \mathcal{H}^2$. Write

$$\varphi(s; z) := \Phi(t-s)z, \ \ s \in [0, t]. \quad (2.19)$$

Then by (2.19), (2.1) and (1.1) (with $u=0$), we see that $\varphi(\cdot;z)$ satisfies

$$\varphi'(s;z)+A\varphi(s;z)+\int_s^t M(\tau-s)\varphi(\tau;z)d\tau=0,\quad s\in(0,t);\quad \varphi(t;z)=z. \tag{2.20}$$

By (1.1) and (2.20), we find

$$\begin{aligned}
&\big\langle y(t),\varphi(t;z)\big\rangle_{L^2(\Omega)}-\big\langle y_0,\varphi(0;z)\big\rangle_{L^2(\Omega)}\\
&=\int_0^t \frac{d}{ds}\big\langle y(s;y_0,u),\varphi(s;z)\big\rangle_{L^2(\Omega)}ds \qquad\qquad (2.21)\\
&=\int_0^t \Big\langle Ay(s)+\int_0^s M(s-\tau)y(\tau)d\tau+\chi_Q u(s),\varphi(s;z)\Big\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}ds+\int_0^t \big\langle y(s),\varphi'(s;z)\big\rangle_{L^2(\Omega)}ds.
\end{aligned}$$

Meanwhile, by the Fubini Theorem, it follows that

$$\begin{aligned}
\int_0^t \Big\langle \int_0^s M(s-\tau)y(\tau)d\tau,\varphi(s;z)\Big\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}ds&=\int_0^t\int_\tau^t M(s-\tau)\big\langle y(\tau),\varphi(s;z)\big\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}dsd\tau\\
&=\int_0^t \Big\langle y(s),\int_s^t M(\tau-s)\varphi(\tau;z)d\tau\Big\rangle_{L^2(\Omega)}ds.
\end{aligned}$$

This, along with (2.21), yields

$$\begin{aligned}
&\big\langle y(t),\varphi(t;z)\big\rangle_{L^2(\Omega)}-\big\langle y_0,\varphi(0;z)\big\rangle_{L^2(\Omega)}\\
&=\int_0^t \Big\langle y(s),\varphi'(s;z)+A\varphi(s;z)+\int_s^t M(\tau-s)\varphi(\tau;z)d\tau\Big\rangle_{L^2(\Omega)}ds+\int_0^t\langle\chi_Q u(s),\varphi(s;z)\rangle_{L^2(\Omega)}ds.
\end{aligned}$$

The above, together with (2.20) and (2.19), indicates

$$\big\langle y(t),z\big\rangle_{L^2(\Omega)}-\big\langle y_0,\Phi(t)z\big\rangle_{L^2(\Omega)}=\int_0^t\langle\chi_Q u(s),\Phi(t-s)z\rangle_{L^2(\Omega)}ds.$$

Since z was arbitrarily taken from $\mathcal{H}^2$, we can use a standard density argument in the above, as well as the first equality in (2.8), to get (2.18). This ends the proof of Proposition 2.6. □

3 Observability estimates of the flow

The main result of this section is the following Theorem 3.1 which gives observability estimates for the flow $\Phi(t)$. This theorem will be used in the proof of Theorem 1.1. *As what we mentioned in the note* (a5) *in Section 1, we are able to get such estimates only for the case where M is real analytic.*

Theorem 3.1. *Assume that (Q,T) satisfies MCC. Then for each $S\geq T$, there is $C>0$ so that for each $z\in L^2(\Omega)$,*

$$\|z\|_{\mathcal{H}^{-4}}\leq C\int_0^S (S-t)^3\|\chi_Q\Phi(S-t)z\|_{L^2(\Omega)}dt; \tag{3.1}$$

$$\int_0^S (S-t)^3\|\chi_Q\Phi(S-t)z\|_{L^2(\Omega)}dt\leq C\|z\|_{\mathcal{H}^{-4}}. \tag{3.2}$$

We will adopt the well-known three-steps strategy for the observability of the wave equation (see, for instant, [27, 1]) to prove Theorem 3.1. To this end, we need several lemmas. The first one is about an intrinsic property of MCC.

Lemma 3.2. *Assume that (Q,T) satisfies MCC. Then for each $S \geq T$,*

$$C(M,Q,T,S) := \inf_{x\in\overline{\Omega}} \int_0^S \chi_Q(t,x)(S-t)^3|M(S-t)|dt > 0. \tag{3.3}$$

Proof. First of all, we mention that Q is open in $\overline{\mathbb{R}^+} \times \overline{\Omega}$ which is viewed as a topology space induced the topology from $\mathbb{R} \times \mathbb{R}^n$. Next, it follows by the definition of MCC for (Q,T) that for each $x \in \overline{\Omega}$, there exists $r_x > 0$ and an interval $I_x \subset\subset (0,T)$ so that

$$I_x \times \Big(B(x,r_x) \cap \overline{\Omega}\Big) \subset Q \cap \big((0,T)\times\overline{\Omega}\big). \tag{3.4}$$

(Here and in what follows, the set $B(x,r)$, with $x \in \mathbb{R}^n$ and $r > 0$, denotes the open ball, centered at x and of radius $r > 0$.) Meanwhile, it is clear that

$$\overline{\Omega} \subset \cup_{x\in\overline{\Omega}} B(x,r_x).$$

Since $\overline{\Omega}$ is compact, the above, together with (3.4), implies that there are finitely many $x_j \in \overline{\Omega}$, $j = 1,\ldots,N$, so that

$$\overline{\Omega} \subset \cup_{j=1}^N B(x_j,r_{x_j}) \ \text{ and } \ \cup_{j=1}^N I_{x_j} \times \Big(B(x_j,r_{x_j}) \cap \overline{\Omega}\Big) \subset Q \cap \big((0,T)\times\overline{\Omega}\big). \tag{3.5}$$

Then for each $\hat{x} \in \overline{\Omega}$, there is $\hat{j} \in \{1,\ldots,N\}$ so that

$$\hat{x} \in B(x_{\hat{j}}, r_{x_{\hat{j}}}) \ \text{ and } \ I_{x_{\hat{j}}} \times \{\hat{x}\} \subset Q \cap \big((0,T)\times\overline{\Omega}\big).$$

These yield that when $S \geq T$ and $\hat{x} \in \overline{\Omega}$,

$$\int_0^S \chi_Q(t,\hat{x})(S-t)^3|M(S-t)|dt \geq \int_{I_{x_{\hat{j}}}} (S-t)^3|M(S-t)|dt,$$

which leads to

$$C(M,Q,T,S) \geq \min_{1\leq j\leq N} \int_{I_{x_j}} (S-t)^3|M(S-t)|dt, \ \text{ when } S \geq T. \tag{3.6}$$

Meanwhile, it follow by the assumption $(\mathfrak{H})$ that $M(t) \neq 0$ for a.e. $t > 0$. This, along with (3.6), shows (3.3). Hence, we finish the proof of Lemma 3.2. ∎

The following lemma corresponds to the first step of the above-mentioned three-steps strategy, which is about a relaxed observability inequality.

Lemma 3.3. *Assume that (Q,T) satisfies MCC. Then for each $S \geq T$, there is $C > 0$ so that*

$$\|z\|_{\mathcal{H}^{-4}} \leq C\bigg(\int_0^S (S-t)^3\|\chi_Q \varPhi(S-t)z\|_{L^2(\Omega)}dt + \|z\|_{\mathcal{H}^{-6}}\bigg) \ \textit{for each } \ z \in L^2(\Omega). \tag{3.7}$$

Proof. Arbitrarily fix $S \geq T$ and $z \in L^2(\Omega)$. By Corollary 2.4 (more precisely, by (2.9) and (2.10)), using the triangle inequality and the fact that $\|A^{-3}z\|_{L^2(\Omega)} = \|z\|_{\mathcal{H}^{-6}}$, we can find $C_1 > 0$ (independent of z) so that

$$\|\chi_Q \Phi(S-t)z\|_{L^2(\Omega)} \;\geq\; \|\chi_Q M(S-t)A^{-2}z\|_{L^2(\Omega)} - C_1(S-t)^{-3}\|z\|_{\mathcal{H}^{-6}} \text{ for each } t \in (0,S).$$

This yields

$$\begin{aligned}&\int_0^S (S-t)^3\|\chi_Q M(S-t)A^{-2}z\|_{L^2(\Omega)}dt\\ &\leq \int_0^S (S-t)^3\|\chi_Q \Phi(S-t)z\|_{L^2(\Omega)}dt + C_1 S\|z\|_{\mathcal{H}^{-6}}.\end{aligned} \tag{3.8}$$

Next, we simply write

$$f(t,x) := \chi_Q(t,x)(S-t)^3 M(S-t)(A^{-2}z)(x), \quad t \in (0,S), \; x \in \Omega. \tag{3.9}$$

The above f can be treated as a function from Ω to $L^1(0,S)$ or from $(0,S)$ to $L^2(\Omega)$. Moreover, we have

$$\begin{aligned}\|f\|_{L^2(\Omega;L^1(0,S))} &= \sup_{\|g\|_{L^2(\Omega)}\leq 1}\int_\Omega \Big(\int_0^S |f(t,x)|dt\Big)g(x)dx\\ &\leq \sup_{\|g\|_{L^2(\Omega)}\leq 1}\int_0^S\int_\Omega |f(t,x)||g(x)|dxdt\\ &\leq \|f\|_{L^1(0,S;L^2(\Omega))}.\end{aligned}$$

This, along with (3.9), indicates

$$\begin{aligned}\int_0^S (S-t)^3\|\chi_Q M(S-t)A^{-2}z\|_{L^2(\Omega)}dt &\geq \Big\|\Big(\int_0^S \chi_Q(S-t)^3|M(S-t)|dt\Big)A^{-2}z\Big\|_{L^2(\Omega)}\\ &\geq \inf_{x\in\Omega}\Big(\int_0^S \chi_Q(S-t)^3|M(S-t)|dt\Big)\|A^{-2}z\|_{L^2(\Omega)}.\end{aligned}$$

By the above, Lemma 3.2 and (1.4), we can find $C_2 > 0$ (independent of z) so that

$$\int_0^S (S-t)^3\|\chi_Q M(S-t)A^{-2}z\|_{L^2(\Omega)}dt \geq C_2\|A^{-2}z\|_{L^2(\Omega)} = C_2\|z\|_{\mathcal{H}^{-4}}.$$

This, together with (3.8), leads to (3.7) and ends the proof of Lemma 3.3. □

The next lemma gives a unique continuation property for the flow $\Phi(\cdot)$. It corresponds to the second step in the above-mentioned three-steps strategy.

Lemma 3.4. *Assume that (Q,T) satisfies MCC. If for some $S \geq T$ and $z \in \mathcal{H}^{-4}$,*

$$\Phi(S-\cdot)z = 0 \;\; \textit{over} \;\; Q \cap \big((0,S)\times\Omega\big), \tag{3.10}$$

then $z = 0$.

Proof. Arbitrarily fix $S \geq T$ and $z \in \mathcal{H}^{-4}$ so that (3.10) holds. First, since $z \in \mathcal{H}^{-4}$, it follows from Corollary 2.4 that

$$\Phi(\cdot)z \in C(\mathbb{R}^+; L^2(\Omega)). \tag{3.11}$$

Next, we let $\{x_j\}_{j=1}^N$, $\big\{B(x_j, r_{x_j})\big\}_{j=1}^N$ and $\{I_{x_j}\}_{j=1}^N$ be given by (3.5) in the proof of Lemma 3.2. From (3.10) and the second conclusion in (3.5), we see that for each $j \in \{1, \dots, N\}$,

$$\Phi(S-t)z = 0 \ \text{ over } \ B(x_j, r_{x_j}) \cap \Omega, \quad t \in I_{x_j}. \tag{3.12}$$

Meanwhile, since M is real analytic over $\overline{\mathbb{R}^+}$, it follows by Proposition 2.3 that $\Phi(\cdot)$ is real analytic from $\mathbb{R}^+$ to $\mathcal{L}(\mathcal{H}^{-4})$. This, along with (3.12), yields that for each $j \in \{1, \dots, N\}$,

$$\Phi(t)z = 0 \ \text{ over } \ B(x_j, r_{x_j}) \cap \Omega \ \text{ for each } \ t > 0.$$

From the above and the first conclusion in (3.5), we see

$$\Phi(t)z = 0 \ \text{ over } \ \Omega \ \text{ for each } \ t > 0.$$

This, along with (3.11), implies

$$\Phi(t)z = 0 \ \text{ in } \ L^2(\Omega) \ \text{ for each } \ t > 0. \tag{3.13}$$

At the same time, since $z \in \mathcal{H}^{-4}$ and $\{e^{tA}\}_{t\geq 0}$ is a C_0 semigroup over $\mathcal{H}^{-4}$, it follows by (2.8) that $\Phi(\cdot)z$ is continuous from $[0, \infty)$ to $\mathcal{H}^{-4}$. This, along with (3.13), yields $z = 0$. Thus, we finish the proof of Lemma 3.4. □

We are now in the position to prove Theorem 3.1, through using the compact-uniqueness argument, as well as Lemmas 3.3-3.4.

Proof of Theorem 3.1. Arbitrarily fix $S \geq T$. First of all, (3.2) is a direct consequence of Corollary 2.4. (Here, a triangle inequality is used.) Next, we will use the well-known compact-uniqueness argument to show (3.1). By contradiction, we suppose that (3.1) was not true. Then there would be $\{\hat{z}_k\}_{k=1}^\infty \subset L^2(\Omega)$ so that

$$\|\hat{z}_k\|_{\mathcal{H}^{-4}} = 1 \ \text{ for each } \ k \in \mathbb{N}^+; \quad \lim_{k\to\infty} \int_0^S (S-t)^3 \|\chi_Q \Phi(S-t) z_k\|_{L^2(\Omega)} dt = 0. \tag{3.14}$$

From the first equality in (3.14), we can find a subsequence of $\{\hat{z}_k\}_{k=1}^\infty$, still denoted in the same manner, and $\hat{z} \in \mathcal{H}^{-4}$ so that

$$\hat{z}_k \to \hat{z} \ \text{ weakly in } \ \mathcal{H}^{-4}, \ \text{ as } \ k \to \infty. \tag{3.15}$$

Then by the second equality in (3.14), it follows that

$$\Phi(S-\cdot)\hat{z} = 0 \ \text{ over } \ Q \cap \big((0, S) \times \Omega\big).$$

Since $\hat{z} \in \mathcal{H}^{-4}$, we can apply Lemma 3.4 to obtain that $\hat{z} = 0$ in $\mathcal{H}^{-4}$. This, together with (3.15), implies that $\lim\limits_{k\to\infty} \|\hat{z}_k\|_{\mathcal{H}^{-6}} = 0$. Then by Lemma 3.3, as well as the second equality in (3.14), we find that $\lim\limits_{k\to\infty} \|\hat{z}_k\|_{\mathcal{H}^{-4}} = 0$, which contradicts the first equality in (3.14). Hence, (3.1) is true. This completes the proof of of Theorem 3.1. □

4 Proofs of main theorems

We will prove Theorem 1.1 and Theorem 1.2 in separate subsections.

4.1 Proof of Theorem 1.1

To prove Theorem 1.1, we need the next Lemma 4.1.

Lemma 4.1. *Assume that (Q,T) satisfies MCC. Then for each $S \geq T$ and each $y_0 \in L^2(\Omega)$,*

$$\mathcal{H}^4 \subset \mathcal{R}_M^\infty(S, y_0). \tag{4.1}$$

Proof. Arbitrarily fix $S \geq T$, $y_0 \in L^2(\Omega)$ and $y_1 \in \mathcal{H}^4$. Let

$$\hat{y}_1 := y_1 - y(S; y_0, 0) = y_1 - \Phi(S)y_0. \tag{4.2}$$

Then by Corollary 2.4, we see

$$\hat{y}_1 \in \mathcal{H}^4. \tag{4.3}$$

Define a functional on a subspace of $L^1(0,S;L^2(\Omega))$ in the manner:

$$\mathcal{F}\Big(\chi_Q \Phi(S-\cdot)z\big|_{(0,S)}\Big) := \langle \hat{y}_1, z\rangle_{L^2(\Omega)}, \quad z \in L^2(\Omega). \tag{4.4}$$

According to (4.3), (4.4) and (3.1) (in Theorem 3.1), $\mathcal{F}$ is well defined and linear and satisfies the estimate:

$$\begin{aligned} \Big|\mathcal{F}\Big(\chi_Q \Phi(S-\cdot)z\big|_{(0,S)}\Big)\Big| &\leq \|\hat{y}_1\|_{\mathcal{H}^4}\|z\|_{\mathcal{H}^{-4}} \\ &\leq C\|\hat{y}_1\|_{\mathcal{H}^4}\|\chi_Q\Phi(S-\cdot)z\|_{L^1(0,S;L^2(\Omega))}, \ \text{ when } \ z \in L^2(\Omega), \end{aligned}$$

for some $C > 0$ independent of z, i.e., $\mathcal{F}$ is bounded. Thus, we can use the Hahn-Banach theorem and the Riesz representation theorem to get $u_0 \in L^\infty(0,S;L^2(\Omega))$ so that

$$\mathcal{F}\Big(\chi_Q \Phi(S-\cdot)z\big|_{(0,S)}\Big) = \int_0^S \big\langle \chi_Q\Phi(S-t)z, u_0(t)\big\rangle_{L^2(\Omega)} dt, \ \ z \in L^2(\Omega).$$

Write $\tilde{u}_0$ for the zero extension of u_0 over $\mathbb{R}^+$. The above, together with Proposition 2.6, the first equality in (2.8) and (4.4), yields

$$\begin{aligned} \langle y(S;0,\tilde{u}_0), z\rangle_{L^2(\Omega)} &= \Big\langle \int_0^S \Phi(S-t)(\chi_Q u_0)(t)dt, z\Big\rangle_{L^2(\Omega)} = \int_0^S \big\langle u_0(t), \chi_Q\Phi(S-t)^* z\big\rangle_{L^2(\Omega)} dt \\ &= \int_0^S \big\langle u_0(t), \chi_Q \Phi(S-t)z\big\rangle_{L^2(\Omega)} dt = \langle \hat{y}_1, z\rangle_{L^2(\Omega)}, \ \text{ when } z \in L^2(\Omega), \end{aligned}$$

from which, it follows that $y(S;0,\tilde{u}_0) = \hat{y}_1$. This, along with (4.2), shows that $y(S;y_0,\tilde{u}_0) = y_1$, which leads to (4.1).

Hence we finish the proof of Lemma 4.1. $\square$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First of all, we recall that $y(\cdot; y_0, u)$ denotes the solution to (1.1), while $z(\cdot; y_0, u)$ denotes the solution to (1.1) where $M = 0$. Arbitrarily fix $1 \le p \le +\infty$, $S \ge T$ and $y_0 \in L^2(\Omega)$. We aim to show (1.6), i.e.,

$$\mathcal{R}_M^p(S, y_0) = \mathcal{R}_0^p(S, y_0) + \mathcal{H}^4. \tag{4.5}$$

The proof is organized by several steps.

Step 1. We show that for each $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$,

$$f_u(\cdot) \in C(\mathbb{R}^+; \mathcal{H}^4), \tag{4.6}$$

where

$$f_u(t) := y(t; y_0, u) - z(t; y_0, u), \ \ t > 0. \tag{4.7}$$

To this end, we arbitrarily fix $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$. Then by (4.7), Proposition 2.6 and Corollary 2.5, we find

$$f_u(t) = \widetilde{\mathcal{R}}_c(t, -A)A^{-2}y_0 + \int_0^t \widetilde{\mathcal{R}}_c(t-s, -A)A^{-2}(\chi_Q u)(s)ds, \ \ t > 0. \tag{4.8}$$

Meanwhile, it follows by Corollary 2.5 that

$$\widetilde{\mathcal{R}_c} \in C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^0)) \cap L^\infty_{loc}(\overline{\mathbb{R}^+}; \mathcal{L}(\mathcal{H}^0)).$$

This, along with (4.8), yields that for each $t_2 \ge t_1 > 0$,

$$\begin{aligned}\|f_u(t_1) - f_u(t_2)\|_{\mathcal{H}^4} \le & \|\widetilde{\mathcal{R}}_c(t_1, -A) - \widetilde{\mathcal{R}}_c(t_2, -A)\|_{\mathcal{L}(\mathcal{H}^0)}\|y_0\|_{\mathcal{H}^0} + \|\widetilde{\mathcal{R}}_c(\cdot, -A)\|_{L^\infty(0,t_2;\mathcal{L}(\mathcal{H}^0))} \\ & \times \Big(\int_0^{t_1} \|u(t_1 - s) - u(t_2 - s)\|_{\mathcal{H}^0}ds + \int_{t_1}^{t_2} \|u(t_2 - s)\|_{\mathcal{H}^0}ds\Big),\end{aligned}$$

which leads to (4.6).

Step 2. We show

$$\mathcal{R}_M^p(S, y_0) \subset \mathcal{R}_0^p(S, y_0) + \mathcal{H}^4. \tag{4.9}$$

Arbitrarily fix $y_1 \in \mathcal{R}_M^p(S, y_0)$. By (1.2), there is $u_1 \in L^p(\mathbb{R}^+; L^2(\Omega))$ so that

$$y_1 = y(S; y_0, u_1) = z(S; y_0, u_1) + \Big(y(S; y_0, u_1) - z(S; y_0, u_1)\Big).$$

Since $z(S; y_0, u_1) \in \mathcal{R}_0^p(S, y_0)$, the above, along with (4.6), leads to (4.9).

Step 3. We show

$$\mathcal{R}_M^p(S, y_0) \supset \mathcal{R}_0^p(S, y_0) + \mathcal{H}^4. \tag{4.10}$$

Arbitrarily fix $\hat{y}_1 \in \mathcal{R}_0^p(S, y_0)$ and $\hat{y}_2 \in \mathcal{H}^4$. According to the definition of $\mathcal{R}_0^p(S, y_0)$ (see (1.2) with $M = 0$), there is $\hat{u}_1 \in L^p(\mathbb{R}^+; L^2(\Omega))$ so that

$$\hat{y}_1 = z(S; y_0, \hat{u}_1). \tag{4.11}$$

Since $\hat{y}_2 \in \mathcal{H}^4$, we see from (4.6) that

$$\hat{y}_3 := \hat{y}_2 - \Big(y(S; y_0, \hat{u}_1) - z(S; y_0, \hat{u}_1)\Big) \in \mathcal{H}^4. \tag{4.12}$$

Since $\hat{y}_3 \in \mathcal{H}^4$, we can apply Lemma 4.1 (where $y_0 = 0$) to find a control $\hat{u}_2 \in L^\infty(\mathbb{R}^+; L^2(\Omega))$, with $\hat{u}_2|_{(S,+\infty)} = 0$, so that $\hat{y}_3 = y(S; 0, \hat{u}_2)$. This, together with (4.12) and (4.11), yields

$$y(S; y_0, \hat{u}_1 + \hat{u}_2) = y(S; y_0, \hat{u}_1) + \hat{y}_3 = \hat{y}_2 + z(S; y_0, \hat{u}_1) = \hat{y}_2 + \hat{y}_1.$$

Since $\hat{y}_1, \hat{y}_2$ were arbitrarily taken from $\mathcal{R}_0^p(S, y_0)$ and $\mathcal{H}^4$ respectively, the above leads to (4.10).

Step 4. From (4.9) *and* (4.10)*,* (4.5) *follows at once.*

Hence, we finish the proof of Theorem 1.1. □

4.2 Proof of Theorem 1.2

To show Theorem 1.2, we need the next Lemma 4.2, which gives an estimate about the heat equation. Since we have not found it in publications, we give its detailed proof for the sake of the completeness of the paper.

Lemma 4.2. *Let $B(x_0, r) \subset \Omega$. Then for each $s \in \mathbb{R}$, there exists $C = C(s) > 0$ so that*

$$\sup_{t \ge 0} \|e^{tA} z\|_{L^2(\Omega \setminus B(x_0,r))} \le C\|z\|_{\mathcal{H}^s} \ \text{ for each } \ z \in \mathcal{H}^s\big(B(x_0, r/2)\big), \tag{4.13}$$

where

$$\mathcal{H}^s\big(B(x_0, r/2)\big) := \big\{z \in \mathcal{H}^s \ : \ supp\, z \subset B(x_0, r/2)\big\}.$$

Proof. It suffices to prove (4.13) for each $s = -m$ with $m \in \mathbb{N}^+$. For this purpose, we arbitrarily fix $z \in \mathcal{H}^{-m}\big(B(x_0, r/2)\big)$. Choose $\{r_l\}_{l=1}^{2m} \subset \mathbb{R}^+$ so that

$$r/2 < r_1 < \cdots < r_{2m} < r. \tag{4.14}$$

Then we take a sequence of functions $\{\rho_l\}_{l=1}^{2m} \subset C_0^\infty(\mathbb{R}^n)$ so that for each $l \in \{1, \cdots, 2m\}$,

$$\rho_l = 0 \ \text{ over } \ B(x_0, r_{l-1}) \ \text{ and } \ \rho_l = 1 \ \text{ over } \ \mathbb{R}^n \setminus B(x_0, r_l). \tag{4.15}$$

Define a sequence of functions $\{f_l\}_{l=0}^{2m}$ in the following manner:

$$f_0(t) := e^{tA} z, \ t > 0; \quad f_l(t) := \rho_l e^{tA} z, \ t > 0, \ l \in \{1, \cdots, 2m\}. \tag{4.16}$$

The rest of the proof is organized by several steps.

Step 1. We prove that for each $\chi_1 \in C_0^\infty(\Omega)$*,* $\chi_2 \in C_0^\infty(\Omega;\mathbb{R}^n)$ *and* $\alpha \geq 0$*, there is* $C = C(\chi_1,\chi_2,\alpha) > 0$ *so that*

$$\|\chi_1 g\|_{\mathcal{H}^{-\alpha-1}} + \|\chi_2 \cdot \nabla g\|_{\mathcal{H}^{-\alpha-1}} \leq C\|g\|_{\mathcal{H}^{-\alpha}}, \ \textit{when}\ g \in \mathcal{H}^{-\alpha}. \tag{4.17}$$

Arbitrarily fix $\chi_1 \in C_0^\infty(\Omega)$ and $\chi_2 \in C_0^\infty(\Omega;\mathbb{R}^n)$. We claim that for each $\alpha \geq 0$, there is $C_1 = C_1(\chi_1,\chi_2,\alpha) > 0$ so that

$$\|\chi_1 f\|_{\mathcal{H}^{\alpha}} + \|\mathrm{div}\,(\chi_2 f)\|_{\mathcal{H}^{\alpha}} \leq C_1\|f\|_{\mathcal{H}^{\alpha+1}}, \ \text{when}\ f \in \mathcal{H}^{\alpha+1}. \tag{4.18}$$

When (4.18) is proved, (4.17) follows by the standard duality argument.

By the interpolation theorem in [20, Theorem 5.1], we see that in order to show (4.18), it suffices to prove (4.18) for $\alpha = 2k$ with $k \in \mathbb{N}$. To this end, we arbitrarily fix $\alpha = 2k$ (with $k \in \mathbb{N}$) and $f \in \mathcal{H}^{2k+1}$. Since χ_1 and χ_2 are compactly supported in Ω, we have

$$\Omega_1 := \mathrm{supp}\,\chi_1 \cup \mathrm{supp}\,\chi_2 \subset\subset \Omega. \tag{4.19}$$

We claim that there is $C_2 > 0$ (independent of f) so that

$$\|f\|_{H^{2k+1}(\Omega_1)} \leq C_2\|f\|_{\mathcal{H}^{2k+1}}. \tag{4.20}$$

In fact, given $h \in \mathcal{H}^{2k+1}$, we have that $A^k h \in \mathcal{H}^1$. From this, (2.2) and (1.4), we see

$$\Delta^k h \in \mathcal{H}^1 = H_0^1(\Omega).$$

Since Δ^k is an elliptic operator of order $2k$, the above yields that $h \in H^{2k+1}_{loc}(\Omega)$ (see for instance [17, Theorem 18.1.29]). Consequently, we have $h|_{\Omega_1} \in H^{2k+1}(\Omega_1)$. Thus, we can define a linear map $\mathcal{T}$ from $\mathcal{H}^1$ to $H^{2k+1}(\Omega_1)$ in the following manner:

$$\mathcal{T}(A^k\bar{h}) := \bar{h}|_{\Omega_1}, \ \ \bar{h} \in \mathcal{H}^{2k+1}. \tag{4.21}$$

By using the closed graph theorem to $\mathcal{T}$, one can obtain that it is bounded. Then by (4.21), there is $C_3 > 0$ so that

$$\|\bar{h}\|_{H^{2k+1}(\Omega_1)} \leq C_3\|A^k\bar{h}\|_{\mathcal{H}^1} = C_3\|\bar{h}\|_{\mathcal{H}^{2k+1}} \ \text{ for each } \ \bar{h} \in \mathcal{H}^{2k+1},$$

which leads to (4.20).

Now, by (1.4), (2.2) and (4.19), there is $C_4 > 0$ (independent of f) so that

$$\|\chi_1 f\|_{\mathcal{H}^{2k}} + \|\mathrm{div}\,(\chi_2 f)\|_{\mathcal{H}^{2k}} = \|\Delta^k(\chi_1 f)\|_{L^2(\Omega)} + \|\Delta^k \mathrm{div}\,(\chi_2 f)\|_{L^2(\Omega)} \leq C_4\|f\|_{H^{2k+1}(\Omega_1)}.$$

The above, along with (4.20), yields (4.18) with $\alpha = 2k$. This ends the proof of Step 1.

Step 2. We prove that for each $l \in \{1,\cdots,2m\}$*, there exists* $C_l > 0$ *(independent of* z*) so that*

$$\|f_l\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{l/2-m})} \leq C_l\|f_{l-1}\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{(l-1)/2-m})}. \tag{4.22}$$

We will show (4.22) by the induction. To this end, we first show (4.22) with $l = 1$. Indeed, from (4.16), one has that $f_1 = \rho_1 f_0$. This, along with (2.2), yields

$$\frac{d}{dt}f_1(t) - Af_1(t) = F_1(t),\ t > 0;\ f_1(0) = 0, \tag{4.23}$$

where

$$F_1(t) := (-\Delta\rho_1)f_0 - 2\nabla\rho_1 \cdot \nabla f_0,\ t > 0. \tag{4.24}$$

Meanwhile, it follows from (4.15) that

$$\Delta\rho_1 \in C_0^\infty(\Omega) \ \text{ and } \ \nabla\rho_1 \in C_0^\infty(\Omega; \mathbb{R}^n).$$

From these and (4.24), we can apply (4.17) (with $(\chi_1, \chi_2, \alpha) = (\Delta\rho_1, \nabla\rho_1, m)$) to find $\widehat{C}_1 > 0$ (independent of z) so that

$$\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})} \le \widehat{C}_1\|f_0\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-m})}. \tag{4.25}$$

We now claim that there is $\widehat{C}_2 > 0$ (independent of z) so that

$$\|f_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{1/2-m})} \le \widehat{C}_2\|f_0\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-m})}. \tag{4.26}$$

Indeed, from (4.23), we can find $C > 0$ (independent of z) so that for each $t > 0$,

$$\begin{aligned}
\|f_1(t)\|_{\mathcal{H}^{1/2-m}} &\le \int_0^t \|e^{(t-s)A}\|_{\mathcal{L}(\mathcal{H}^{-1-m};\mathcal{H}^{1/2-m})}\|F_1(s)\|_{\mathcal{H}^{-1-m}}ds \\
&\le \left(\int_0^t \left\|\left[(-A)^{3/4}e^{(t-s)A/2}\right]e^{(t-s)A/2}\right\|_{\mathcal{L}(\mathcal{H}^{-1-m})}ds\right)\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})} \\
&\le \left(\int_0^t C(t-s)^{-3/4}e^{-(t-s)\eta_1/2}ds\right)\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})}.
\end{aligned}$$

This, along with (4.25), leads to (4.26). So (4.22) holds for $l = 1$.

Next, we assume that for some $l_0 \in \{1, \cdots, 2m-1\}$, (4.22) holds for all $l \le l_0$. We aim to prove (4.22) with $l = l_0 + 1$. In fact, from (4.16) and (4.15), we have $f_{l_0+1} = \rho_{l_0+1}f_{l_0}$. By this and using the similar way to that used in the proof of (4.22) with $l = 1$, we can get (4.22) with $l = l_0 + 1$. This ends the proof of Step 2.

Step 3. We verify (4.13).

Since $z \in \mathcal{H}^{-m}$, it follows from (4.16) and (1.5) that

$$\|f_0(t)\|_{\mathcal{H}^{-m}} = \|e^{tA}z\|_{\mathcal{H}^{-m}} \le \|z\|_{\mathcal{H}^{-m}},\ t \ge 0.$$

This, together with (4.22) and (4.14)-(4.16), yields (4.13) with $s = -m$.

Hence, we finish the proof of Lemma 4.2. $\square$

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. By contradiction, we suppose that (1.7) holds, but there is $\hat{x} \in \overline{\Omega}$ so that

$$(t, \hat{x}) \notin \overline{Q} \ \text{ for each } \ t \in [0, T].$$

Then one can find $B(x_0, r) \subset \Omega$ so that

$$Q \cap \big([0,T] \times \Omega\big) \subset [0,T] \times \big(\Omega \setminus B(x_0, r)\big). \tag{4.27}$$

The rest of the proof is divided into the following two steps:

Step 1. We show that there is $C > 0$ so that

$$\|\Phi(T)z\|_{L^2(\Omega)} \leq C\|\Phi(T - \cdot)z\|_{L^\infty(0,T;L^2(\Omega\setminus B(x_0,r)))} \ \textit{for each} \ z \in L^2(\Omega). \tag{4.28}$$

We first claim that under the assumption (1.7), the system (1.1) is L^p-null controllable with a cost over $[0, T]$, i.e., there is $C > 0$ so that for each $y_0 \in L^2(\Omega)$, there is $u_{y_0} \in L^p(0, T; L^2(\Omega))$ satisfying

$$y(T; y_0, \tilde{u}_{y_0}) = 0 \ \text{ and } \ \|u_{y_0}\|_{L^p(0,T;L^2(\Omega))} \leq C\|y_0\|_{L^2(\Omega))}. \tag{4.29}$$

(Here and in what follows, given a control v over $[0, T]$, we use $\tilde{v}$ to denote its zero extension over $\mathbb{R}^+$.) Indeed, if we define the operator:

$$L_T(u) := y(T; 0, \tilde{u}), \ \ u \in L^p(0, T; L^2(\Omega)),$$

then by (1.7), we have $\operatorname{Range}\Phi(T) \subset \operatorname{Range}L_T$. Without loss of generality, we can assume that L_T is injective, for otherwise, we can replace L_T by the operator $\widetilde{L}_T$ (from the quotient space $L^p(0, T; L^2(\Omega))/\ker L_T$ to $L^2(\Omega)$) which is uniquely induced by L_T. Then for each $y_0 \in L^2(\Omega)$, there is a unique u_{y_0} so that

$$\Phi(T)y_0 = L_T u_{y_0}, \text{ i.e., } \ y(T; y_0, \tilde{u}_{y_0}) = 0. \tag{4.30}$$

Meanwhile, according to the closed graph theorem, the map $y_0 \mapsto u_{y_0}$, $y_0 \in L^2(\Omega)$ is continuous. which leads to the second inequality in (4.29). This, along with (4.30), shows (4.29).

Next, by (4.29), using the classical duality argument (see for instance [30, Theorem 1.18]), we can obtain the following observability: there is $C > 0$ so that

$$\|\Phi(T)z\|_{L^2(\Omega)} \leq C\|\chi_Q\Phi(T - \cdot)z\|_{L^q(0,T;L^2(\Omega))} \ \text{for each} \ z \in L^2(\Omega), \tag{4.31}$$

where q is such that $1/p + 1/q = 1$.

Finally, (4.28) follows from (4.31) and (4.27).

Step 2. We make a contradiction to (4.28) by choosing a suitable sequence $\{z_k\}_{k\geq 1}$ in $L^2(\Omega)$.

Let $\{h_l\}_{l\in\mathbb{N}}$ be given by (2.3). Then it follows by Theorem 2.1 that $\{h_l(T)\}_{l\geq 1}$ is not the zero sequence. Thus, we have

$$J := \min\Big\{l \in \mathbb{N}^+ \ : \ h_l(T) \neq 0\Big\} < +\infty. \tag{4.32}$$

Meanwhile, we choose $\{\varepsilon_k\}_{k\geq 1}\subset(0,r/2)$ and $\rho\in C_0^\infty(\mathbb{R}^n)$ so that

$$\lim_{k\to\infty}\varepsilon_k=0;\quad \|\rho\|_{L^2(\mathbb{R}^n)}=1;\quad \rho(x)=0,\ |x|_{\mathbb{R}^n}\geq 1. \tag{4.33}$$

Define $\{w_k\}_{k\geq 1}\subset C_0^\infty(\Omega)$ by

$$w_k(x):=\varepsilon_k^{-n/2}\rho\Big(\frac{x-x_0}{\varepsilon_k}\Big),\ x\in\Omega.$$

From this and (4.33), one can directly check

$$\operatorname{supp} w_k\subset B(x_0,r/2),\ \forall\, k\in\mathbb{N}^+;\ \|w_k\|_{L^2(\Omega)}=1,\ \forall\, k\in\mathbb{N}^+;\ \mathrm{w}-\lim_{k\to\infty}w_k=0 \text{ in } L^2(\Omega). \tag{4.34}$$

Now we define the sequence $\{z_k\}_{k\geq 1}\subset L^2(\Omega)$ by

$$z_k:=A^{J+1}w_k,\ \ k\geq 1. \tag{4.35}$$

Next, from (4.35) and (4.34), as well as (2.2), we see that for each $l\in\{0,1,\ldots,J+1\}$,

$$\operatorname{supp}(A^{-l}z_k)\subset B(x_0,r/2),\ \ \forall\, k\in\mathbb{N}^+;\quad \lim_{k\to\infty}z_k=0\ \text{ in }\ \mathcal{H}^{-2J-4}. \tag{4.36}$$

Meanwhile, by Theorem 2.1 with $N=J+1$, we have

$$\begin{aligned}\varPhi(t)=&e^{tA}\Big(Id+\sum_{l=0}^{J}p_l(t)(-A)^{-l-1}\Big)+\sum_{l=0}^{J}h_l(t)(-A)^{-l-1}+\mathcal{R}_{J+1}(t,-A)(-A)^{-J-2}\\ :=&\mathcal{P}(t)+\mathcal{H}(t)+\widetilde{\mathcal{R}}(t),\ \ t>0,\end{aligned} \tag{4.37}$$

where $\mathcal{R}_{J+1}(\cdot,-A)\in C(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^0))\bigcap L^\infty_{loc}(\overline{\mathbb{R}^+};\mathcal{L}(\mathcal{H}^0))$ is given in Theorem 2.1 with $N=J+1$. With regard to three terms on the right hand side of (4.37), we have what follows: First, from the second conclusion in (4.36), as well as the regularity of $\mathcal{R}_{J+1}(\cdot,-A)$, we find

$$\lim_{k\to\infty}\sup_{0\leq t\leq T}\|\widetilde{\mathcal{R}}(t)z_k\|_{L^2(\Omega)}=0. \tag{4.38}$$

Second, by the smoothing effect of $\{e^{tA}\}_{t\geq 0}$, we obtain

$$\lim_{k\to\infty}\mathcal{P}(T)z_k=0\ \text{ in }\ L^2(\Omega). \tag{4.39}$$

Third, by (4.32), we have

$$\mathcal{H}(T)=h_J(T)(-A)^{-J-1},$$

which, along with (4.35) and the second equality in (4.34), yields

$$\|\mathcal{H}(T)z_k\|_{L^2(\Omega)}=|h_J(T)|\neq 0,\ \ \forall\, k\in\mathbb{N}^+. \tag{4.40}$$

Now from (4.37), (4.38), (4.39) and (4.40), it follows that

$$\lim_{k\to\infty}\|\varPhi(T)z_k\|_{L^2(\Omega)}=|h_J(T)|\neq 0. \tag{4.41}$$

Finally, by (4.36) and by the iterative use of Lemma 4.2 (with $z = A^{-l}z_k$, $l \in \{0, \cdots, J+1\}$), we find

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|\mathcal{P}(t)z_k\|_{L^2(\Omega\setminus B(x_0,r))} = 0. \tag{4.42}$$

Meanwhile, from the first conclusion in (4.36) and the definition of $\mathcal{H}(t)$ (see (4.37)), we see that for each $k \in \mathbb{N}^+$,

$$\mathcal{H}(t)z_k = 0 \text{ over } \Omega \setminus B(x_0, r), \quad 0 \le t \le T. \tag{4.43}$$

From (4.37), (4.42), (4.43) and (4.38), we obtain

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|\Phi(T-t)z_k\|_{L^2(\Omega\setminus B(x_0,r))} = 0. \tag{4.44}$$

Now, combining (4.44) and (4.41) contradicts (4.28).

Hence, we complete the proof Theorem 1.2. $\square$

5 Appendix

5.1 Several properties on reachable sets and MCC

Proposition 5.1. *The following conclusions are true:*

(i) The set $\mathcal{R}^p_M(S, y_0)$ is a linear subspace if and only if $0 \in \mathcal{R}^p_M(S, y_0)$;

(ii) If $\mathcal{R}^p_M(S, y_0) = \mathcal{R}^p_0(S, y_0) + \mathcal{H}^4$, then $0 \in \mathcal{R}^p_M(S, y_0)$;

(iii) If MCC holds for (Q, T), then each $\mathcal{R}^p_M(S, y_0)$, with $S \ge T$, is a linear subspace;

(iv) MCC for (Q, T) implies both (1.6) and (1.7);

(v) Either (1.6) or (1.7) implies MCC for $(\overline{Q}, T)$.

Proof. Since $y(S; y_0, u) = y(S; y_0, 0) + y(S; 0, u)$, it follows from (1.2) that $\mathcal{R}^p_M(S, y_0)$ is an affine space. From this, the conclusion (i) follows at once.

We next show (ii). Suppose

$$\mathcal{R}^p_M(S, y_0) = \mathcal{R}^p_0(S, y_0) + \mathcal{H}^4. \tag{5.1}$$

We provide two ways to show that $0 \in \mathcal{R}^p_M(S, y_0)$. The first way is as follows: Recall that $z(\cdot; y_0, u)$ denotes the solution of the controlled heat equation (i.e., (1.1) where $M = 0$). Then by the smooth effect of the heat equation, we have

$$z(S; y_0, 0) \in \mathcal{R}^p_0(S, y_0) \cap \mathcal{H}^4,$$

which, along with (5.1), yields that $0 \in \mathcal{R}^p_M(S, y_0)$. The second way is as follows: By the null controllability of the pure heat equation (see for instance [10, Theorem 3.1]), there is $u \in L^p(\mathbb{R}^+; L^2(\Omega))$ so that $z(S; y_0, u) = 0$. This, along with (5.1), yields that $0 \in \mathcal{R}^p_M(S, y_0)$.

The conclusion (iii) follows from Theorem 1.1 and the above conclusions (ii) and (i).

The conclusion (iv) follows from Theorem 1.1 and the above conclusion (ii).

The conclusion (v) follows from Theorem 1.2 and the above conclusion (ii).

Hence, we finish the proof of Proposition 5.1. $\square$

5.2 An example on MCC

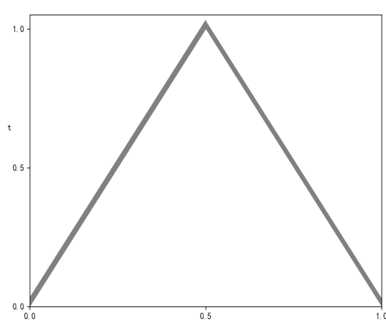
Example 5.2. Let $\Omega := (0, 1)$. Given $\varepsilon \in (0, 1)$, define two functions:

$$f_\varepsilon(x) := \begin{cases} 2x, \ x \in [0, 1/2), \\ -2x + 2, \ x \in [1/2, 1] \end{cases} \quad \text{and} \quad g_\varepsilon(x) := \begin{cases} 2x + \varepsilon, \ x \in [0, 1/2), \\ -2x + 2 + \varepsilon, \ x \in [1/2, 1]. \end{cases}$$

Take the following control region (see the following figure where $\varepsilon = 0.03$):

$$Q := \Big\{(t, x) \in \mathbb{R}^+ \times [0, 1] \ : \ f_\varepsilon(x) < t < g_\varepsilon(x)\Big\}.$$

By direct computations, we can check what follows: (i) For each $T > 1$, the pair (Q, T) satisfies



MCC; (ii) By a suitable choice of ε, the cross sections of Q are allowed to have sufficiently small lengthes; (iii) The pair (Q, T) does not satisfy [4, Assumption 4.1]. Consequently, our MCC is strictly weaker than [4, Assumption 4.1].

Acknowledgments. The first two authors would like to gratefully thank Dr. Huaiqiang Yu for his valuable comments and suggestions.

The first and second authors were supported by the National Natural Science Foundation of China under grants 11971022 and 11801408, respectively.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No.765579-ConFlex. The third author has also been funded by the Alexander von Humboldt-Professorship program, the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon), grant MTM2017-92996 of MINECO (Spain), ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, ICON-ANR-16-ACHN-0014 of the French ANR and Nonlocal PDEs: Analysis, Control and Beyond, AFOSR Grant FA9550-18-1-0242.

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