The turnpike property in nonlinear optimal control — A geometric approach

Noboru Sakamoto [a], Enrique Zuazua [b]

[a]*Faculty of Science and Engineering, Nanzan University, Yamazato-cho 18, Showa-ku, Nagoya, 464-8673, Japan*

[b]*Chair in Applied Analysis, Alexander von Humboldt-Professorship, Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg 91058 Erlangen, Germany*
Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
Chair of Computational Mathematics, Fundación Deusto, Avda Universidades, 24, 48007, Bilbao, Basque Country, Spain

Abstract

This paper presents, using dynamical system theory, a framework for investigating the turnpike property in nonlinear optimal control. First, it is shown that a turnpike-like property appears in general dynamical systems with hyperbolic equilibrium and then, apply it to optimal control problems to obtain sufficient conditions for the turnpike to occurs. The approach taken is geometric and gives insights for the behaviors of controlled trajectories as well as links between the turnpike and stability and/or stabilizability in nonlinear control theory. It also allows us to find simpler proofs for existing results on the turnpike properties. Attempts to remove smallness restrictions for initial and target states are also discussed based on the geometry of (un)stable manifolds and a recent result on exponential stabilizability of nonlinear control systems obtained by one of the authors.

Key words: Optimal control; Nonlinear system; Turnpike.

1 Introduction

The turnpike property was first recognized in the context of optimal growth by economists (see, e.g., [32]). The turnpike theorems say that for a long-run growth, regardless of starting and ending points, it will pay to get into a growth phase, called *von Neumann path*, in the most of intermediate stages. It is exactly like a turnpike and a network of minor roads; "if origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel ..." (quoted from [11]).

In control theory, independently of the turnpike theorems in econometrics, this property was investigated as *dichotomy* in linear optimal control [52,42] and later extended to nonlinear systems [1]. In optimal control, the turnpike property essentially means that the solution of a large finite horizon optimal control problem is determined by the system and cost function and independent of time intervals, initial and terminal conditions except in the thin layers at the beginning and the end of the time interval (see, e.g., [6,55,41]). In the last decades, much progress has been made in the theory of turnpike for finite or infinite dimensional and linear or nonlinear control systems. In [39], the authors study the turnpike for linear finite and infinite dimensional systems and derive a simple but meaningful inequality, which we term *turnpike inequality*. Their works are extended to finite-dimensional nonlinear systems [49],

⋆ This paper was not presented at any IFAC meeting. Corresponding author: Noboru Sakamoto.
a: This work was partially funded by by JSPS KAKENHI Grant Numbers JP26289128, JP19K04446 and by Nanzan University Pache Research Subsidy I-A-2 for 2019 academic year.
b: Supported, in part, by the Alexander von Humboldt-Professorship program, by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon), by grant MTM2017-92996 of MINECO (Spain), by ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, by ICON of the French ANR and Nonlocal PDEs: Analysis, Control and Beyond, and by AFOSR Grant FA9550-18-1-0242.

Email addresses: noboru.sakamoto@nanzan-u.ac.jp (Noboru Sakamoto), enrique.zuazua@fau.de (Enrique Zuazua).

the semi-linear heat equation [40], the wave equation [22,56], periodic turnpike for systems in Hilbert spaces [48], optimal shape design [27], optimal boundary control for hyperbolic systems [21] and general evolution equations [20]. The turnpike property draws attentions of system theory researchers from the viewpoints of model predictive control [17,12], dissipative systems (see, e.g., [53,54]) [4,9,14,18,19], mixed-integer optimal control [15], mechanical systems [13] and the maximum hands-off control [33,45]. The turnpike attracts attentions in control theory since it often reveals an essential structure of the optimal control problem under consideration and leads to a significant simplification or a useful approximation of the problem. From this point of view, one of the issues in investigating the turnpike is to give conditions under which the occurrence of the turnpike is assured. Especially, it is important to know how large initial states and terminal states can be taken for the turnpike to occur. The approach based on dissipative system theory is shown to be effective from these perspectives.

In this paper, we first show that turnpike-like behaviors naturally appear in general dynamical systems with hyperbolic equilibrium. The main technique we use is the λ-lemma which describes trajectory behaviors near invariant manifolds such as stable and unstable manifolds. That the turnpike-like inequality holds implies that if one fixes two ends of a trajectory close to stable and unstable manifolds and designates the time duration sufficiently large, then the trajectory necessarily converges to these manifolds to spend the most of the time near the equilibrium. This is exactly the turnpike property. It should be noted that the two ends, as long as they are close to the manifolds, do not need to lie in the vicinity of the equilibrium. Using this property of hyperbolic dynamical systems, we try to render the problem of determining the domains of initial and terminal states for the turnpike to occur into the size and geometries of the stable and unstable manifolds.

We apply the turnpike-like inequality to a class of optimal control problems in which terminal states are not specified and the steady state optimal solutions are not the origin as in [39,40,56,48] as well as to a class of optimal control problems in which two terminal states are specified and the steady state optimal point is the origin as in [52,1,49]. For both classes of problems, we employ a Dynamic Programming approach with Hamilton-Jacobi equations (HJEs). The characteristic equations for HJEs are Hamiltonian systems and the stabilizability (controllability for the second class) and detectability conditions assure that the equilibrium of the Hamiltonian systems is hyperbolic. The *controlled trajectories* appear in these Hamiltonian systems and we apply the turnpike result for dynamical system. Then, the existence of the trajectory satisfying initial and boundary conditions is guaranteed. In this paper, we derive *sufficient conditions* for optimality by using the Dynamic Programming and HJEs and by imposing a condition that guarantees the existence of the solution to the HJEs (Lagrangian submanifold property, see, e.g., [30, page 93]).

The present manuscript expands upon our conference contribution [46] incorporating a new result on the relationship between nonlinear stabilizability and the existence of infinite horizon optimal control [44]. It allows one to give an estimate of the existence region of a stable manifold of hyperbolic Hamiltonian system associated with an optimal control problem, from which one may be able to predict the occurrence of turnpike (see Section 4.2). This prediction exactly follows the procedure of rendering the turnpike analysis into the geometries of stable and unstable manifolds developed in § 2.The manuscript also contains examples worked out to show how the proposed geometric approach is effectively applied for the turnpike analysis and appendices for necessary results in the theory of algebraic Riccati equations and for stable manifold estimate in Hamiltonian systems.

The organization of the paper is as follows. In Section 2 we review key tools from dynamical system theory and derive the turnpike inequality. In Section 3, we apply it to optimal control problems. Section 3.1 handles the problem where terminal state is free and Section 3.2 handles the problem where initial and terminal states are fixed. Section 4 shows turnpike analyses for a class of nonlinear systems for which target z in (OCP_1) can be taken arbitrarily large and a class of nonlinear systems for which initial states can be taken arbitrarily large. Section 5 discusses possible extensions for more general turnpike using the geometric approach.

2 Turnpike in dynamical systems

Let us consider a nonlinear dynamical system of the form

$$\dot{z} = f(z), \tag{1}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is of C^r class ($r \geqslant 1$). We assume that $f(0) = 0$ and *hyperbolicity of f at* 0, namely, assume that $Df(0) \in \mathbb{R}^{N\times N}$ has k eigenvalues ($0 < k < N$) with strictly negative real parts and $N - k$ eigenvalues with strictly positive real parts.

It is known, as *the stable manifold theorem*, that there exist C^r manifolds S and U, called *stable manifold* and *unstable manifold* of (1) at 0, respectively, defined by

$$S := \{z \in \mathbb{R}^N \mid \varphi(t, z) \to 0 \text{ as } t \to \infty\},$$
$$U := \{z \in \mathbb{R}^N \mid \varphi(t, z) \to 0 \text{ as } t \to -\infty\},$$

where $\varphi(t, z)$ is the solution of (1) starting z at $t = 0$. Let E^s, E^u be stable and unstable subspaces in $\mathbb{R}^N$ of $Df(0)$ with dimension k, $N - k$, respectively. It is known that S, U are invariant under the flow of f and are tangential to E^s, E^u, respectively, at $z = 0$. See, e.g., [23, § III.6] and [35, Chapter 2, § 6] for more details on the theory of stable manifold.

We will consider limiting behavior of submanifolds under the flow of f and need to introduce topology for maps and manifolds. Let M be a compact manifold of dimension m and the space $C^r(M, \mathbb{R}^l)$ of C^r maps, $0 \leqslant r < \infty$, defined on M. There exists a natural vector space structure on $C^r(M, \mathbb{R}^l)$. Since M is compact, we take a finite cover of M by open sets $V_1, \dots, V_k$ and take a local chart (z_i, U_i) for M with $z_i(U_i) = B(2)$ such that $z_i(V_i) = B(1)$, $i = 1, \dots, k$, where $B(1)$ and $B(2)$ are the balls of radius 1 and 2 at the origin of $\mathbb{R}^m$. For a map $g \in C^r(M, \mathbb{R}^l)$, we define a norm by

$$\|g\|_r := \max_i \sup\{|g(u)|, \|Dg^i(u)\|, \dots, \\ \|D^r g^i(u)\| \mid u \in B(1)\},$$

where $g^i = g \circ {z_i}^{-1}$, local representation of g, and $\|\cdot\|$ is a norm for linear maps. It is known that $\|\cdot\|_r$ does not depend on the choice of finite cover (see [35, page 20]) and we call it C^r norm. For maps in $C^r(M, N)$ where N is a manifold, we embed N in a Euclidean space with sufficiently high dimension. Let L, L' be C^r submanifolds of M and let $\varepsilon > 0$. We say that L *and* L' *are* ε C^r*-close* if there exists a C^r diffeomorphism $h : L \to L'$ such that $\|i' \circ h - i\|_r < \varepsilon$, where $i : L \to M$ and $i' : L' \to M$ are inclusion maps. In this case, we use the notation $d^r_L(L') := \|i' \circ h - i\|_r$.

By a k-dimensional (topological) disc we mean a set that is homeomorphic to $D^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 \leqslant 1\}$. The following lemma is known as *the* λ*-lemma* or *inclination lemma* and plays a crucial role in the theory of dynamical systems (see [35, Chapter 2, § 7] and [51, § 5.1]).

Lemma 1 (The λ-lemma) *Suppose that $z = 0$ is a hyperbolic equilibrium for (1). Suppose also that S and U are k, $(N-k)$-dimensional stable and unstable manifolds of f at 0, respectively. For any $(N-k)$-dimensional disc B in U, any point $z \in S$, any $(N-k)$-dimensional disc D transversal to S at z and any $\varepsilon > 0$, there exists a $T > 0$ such that if $t > T$, $\varphi(t, D)$ contains an $(N-k)$-dimensional disc $\tilde{D}$ with $d^1_B(\tilde{D}) < \varepsilon$.*

Next, we show that *the turnpike behavior* appears in the transition of points near the stable manifold to points near the unstable manifold if the transition duration is designated large. Let $z_0 \in S$ and $z_1 \in U$ be arbitrary given points. From the stable manifold theorem, it holds that

$$|\varphi(t, z_0)| < K e^{-\mu t} \text{ for } t \geqslant 0, \tag{2a}$$

$$|\varphi(t, z_1)| < K e^{\mu t} \text{ for } t \leqslant 0, \tag{2b}$$

where $K > 0$ is a constant dependent on z_0 and z_1 and $\mu > 0$ is a constant independent of z_0 and z_1.

Proposition 2 *Suppose that $z = 0$ is a hyperbolic equilibrium for (1). Let S and U be the stable manifold of dimension k and the unstable manifold of dimension $N-k$ at $z = 0$, respectively. Take z_0, z_1, K and μ satisfying (2).*

(i) There exists a $T_0 > 0$ such that for every $T > T_0$ there exists a $\rho = \rho(T) > 0$ such that

$$|\varphi(t, y)| < K e^{-\mu t} \textit{ for } t \in [0, T],\ y \in B(z_0, \rho),$$

where $B(z_0, \rho)$ is the N-dimensional open ball centered at z_0 with radius ρ. Moreover, $\rho \to 0$ as $T \to \infty$.

(ii) There exist a $T_0 < 0$ such that for every $T < T_0$ there exists a $\rho = \rho(T) > 0$ such that

$$|\varphi(t, y)| < K e^{\mu t} \textit{ for } t \in [T, 0],\ y \in B(z_1, \rho).$$

Moreover, $\rho \to 0$ as $T \to -\infty$.

(iii) For any $(N-k)$-dimensional disc $\bar{D}$ transversal to S at z_0 and any k-dimensional disc $\bar{E}$ transversal to U at z_1, there exists a $T_0 > 0$ such that for any $T > T_0$ there exist $(N-k)$-dimensional disc $D \subset \bar{D}$ transversal to S at z_0 and k-dimensional disc $E \subset \bar{E}$ transversal to U at z_1 such that $\varphi(T, D)$ intersects $\varphi(-T, E)$ at a single point.

Proof. (i) Suppose that the first statement is false. Then, there exist a $T > 0$ such that for all $n \in \mathbb{N}$, there exist $t_n \in [0, T]$ and $y_n \in B(z_0, 1/n)$ such that $|\varphi(t_n, y_n)| \geqslant K e^{-\mu t_n}$ holds for $n \in \mathbb{N}$. Taking a subsequence, we may assume that $t_n \to \tau \in [0, T]$ as $n \to \infty$. This implies that $|\varphi(\tau, z_0)| \geqslant K e^{-\mu \tau}$, which contradicts (2a). For the second statement, we prove the following. For any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that for any $T > T_\varepsilon$, if $|\varphi(t, y)| < K e^{-\mu t}$ for $t \in [0, T]$, $y \in B(z_0, \rho)$, then, ρ necessarily satisfies $\rho < \varepsilon$. If this is not true, then, there exist $\varepsilon_0 > 0$ and $\rho \geqslant \varepsilon_0$ such that for any $T > 0$, $|\varphi(t, y)| < K e^{-\mu t}$ for $t \in [0, T]$, $y \in B(z_0, \rho)$. But, this means that $B(z_0, \rho) \subset S$, which contradicts that the dimensions of $B(z_0, \rho)$ and S are N and $k (< N)$, respectively.

(ii) Proof is the same as (i).

(iii) First we take an $(N-k)$-dimensional disc U_0 in U passing through 0, a k-dimensional disc S_0 in S passing through 0 and an $\varepsilon > 0$ arbitrarily. From the λ-lemma, there exists a $T_0 > 0$ such that for any $T > T_0$ there exists an $(N-k)$-dimensional disc $D \subset \bar{D}$ transversal to S at z_0 and a k-dimensional disc $E \subset \bar{E}$ transversal to U at z_1 such that $d^1_{U_0}(\varphi(T, D)) < \varepsilon$, $d^1_{S_0}(\varphi(-T, E)) < \varepsilon$.

Since $E^s \cap E^u = \{0\}$, it is possible to take ε, S_0 and U_0 so that $\varphi(T, D)$ intersects $\varphi(-T, E)$ at a single point. ∎

Remark 1 *It should be noted that the above statements, especially (i) and (ii), are only on finite interval $[0, T]$. This is the major difference from the trajectories on the stable and unstable manifolds.*

Theorem 3 *Suppose that $z = 0$ is a hyperbolic equilibrium for (1). Let S and U be the stable manifold of dimension k and the unstable manifold of dimension $N - k$ at $z = 0$, respectively. For any $z_0 \in S$, any $z_1 \in U$, any $(N - k)$-dimensional disc $\bar{D}$ transversal to S at z_0 and any k-dimensional disc $\bar{E}$ transversal to U at z_1, there exists a $T_0 > 0$ such that for every $T > T_0$ there exist $\rho = \rho(T) > 0$, $y_0 \in B(z_0, \rho) \cap \bar{D}$ and $y_1 \in B(z_1, \rho) \cap \bar{E}$ such that $\varphi(T, y_0) = y_1$ and*

$$|\varphi(t, y_0)| < K \left[e^{-\mu t} + e^{-\mu(T-t)} \right] \text{ for } t \in [0, T].$$

Moreover, $\rho \to 0$ as $T \to \infty$.

Proof. Take the largest T_0 and the smallest ρ in Proposition 2. We rename this T_0 as $T_0/2$. Take arbitrary $T > T_0$ and use Proposition 2-(iii) to get a disc D which is $(N - k)$-dimensional and transversal to S at z_0 and a disc E which is k-dimensional and transversal to U at z_1 satisfying $D \subset B(z_0, \rho)$ and $E \subset B(z_1, \rho)$. This is possible by taking smaller S_0 and U_0 in the proof of Proposition 2-(iii). Then, there exists a single point ζ such that $\varphi(T/2, D) \cap \varphi(-T/2, E) = \{\zeta\}$ (see Fig 1). Let $y_0 := \varphi(-T/2, \zeta)$. Then, $y_0 \in D \subset B(z_0, \rho)$ and by Proposition 2-(i), we have

$$|\varphi(t, y_0)| < K e^{-\mu t} \text{ for } 0 \leqslant t \leqslant T/2. \tag{3}$$

Let $y_1 := \varphi(T/2, z)$. Then, $y_1 \in E \subset B(x_1, \rho)$ and

$$|\varphi(t, y_1)| < K e^{\mu t} \text{ for } -T/2 \leqslant t \leqslant 0.$$

This shows that

$$|\varphi(t + T, y_0)| < K e^{\mu t} \text{ for } -T/2 \leqslant t \leqslant 0,$$

and consequently,

$$|\varphi(t, y_0)| < K e^{-\mu(T-t)} \text{ for } T/2 \leqslant t \leqslant T. \tag{4}$$

Combining (3) and (4), we get the inequality in the theorem. The last assertion follows from Proposition 2-(i) and (ii). ∎

3 Turnpike in nonlinear optimal control

Let us consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u, \; x(t_0) = x_0, \tag{5}$$

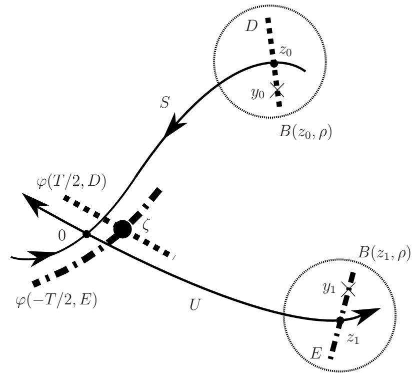

Fig. 1. A scheme of the proof of Theorem 3

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are of C^2 class with $f(0) = 0$, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. Let $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a C^2 function of x and u. An optimal control problem or OCP is to find a control input for (5) such that the cost functional

$$J(u) = \int_0^T L(x(t), u(t)) \, dt$$

is minimized, where we set $J(u) = +\infty$ when the existence domain of solution for (5) is strictly contained in $[0, T)$. There are several types in OCPs depending on whether or not the terminal time T is specified and whether or not the state variables are specified at the terminal time. In this paper, we consider OCPs where the terminal time T is specified and two types of OCPs; one in which the state variables are free at $t = T$ and another in which they are fixed at $t = T$. For both types of OCPs, we are interested in the relationship between the solution u_T and corresponding trajectory x_T of an OCP and steady state optimal pair $(\bar{u}, \bar{x})$, which will be defined more precisely later on.

Definition 1 *[6]*An optimal pair (u_T, x_T) has the *turnpike property* if for any $\varepsilon > 0$, there exists an $\eta_\varepsilon > 0$ such that

$$|\{t \geqslant 0 \,|\, |u_T(t) - \bar{u}| + |x_T(t, x_0) - \bar{x}| > \varepsilon\}| < \eta_\varepsilon$$

for all $T > 0$, where η_ε depends only on ε, f, g, x_0, and L and $|\cdot|$ denotes length (Lebesgue measure) of interval.

Definition 2 An optimal pair (u_T, x_T) satisfies *the (exponential) Turnpike inequality* if x_T and u_T satisfy

$$|u_T(t) - \bar{u}| + |x_T(t, x_0) - \bar{x}| < K \left[e^{-\mu t} + e^{-\mu(T-t)} \right] \tag{6}$$

for $t \in [0, T]$ and for some constants $K > 0$ and $\mu > 0$ independent of T.

Remark 2 Inequality (6) means that when T is large, $u_T(t)$ and $x_T(t, x_0)$ are exponentially close to $\bar{u}$ and $\bar{x}$

for most of the time in $[0,T]$ except at the beginning and the end of $[0,T]$. It is thus sometimes called *exponential* turnpike inequality. It is known that the turnpike inequality is a sufficient condition for the turnpike property in Definition 1. Also, it should be noted that requiring (6) limits ourselves to *the exponential input-state turnpike* defined in [19].

3.1 The OCP with state variables unspecified at the terminal time

For system (5), we consider the following cost functional

$$J_1(u) = \frac{1}{2}\int_0^T |Cx(t) - z|^2 + |u(t)|^2\,dt,$$

where $C \in \mathbb{R}^{r\times n}$ and $z \in \mathbb{R}^r$ is a given vector (*target*). We call this problem $(\mathrm{OCP}_1)_T$;

$(\mathrm{OCP}_1)_T$: Find a control $u \in L^\infty(0,T;\mathbb{R}^m)$ such that $J_1(u)$ along (5) is minimized over all $u \in L^\infty(0,T,\mathbb{R}^m)$.

Associated with $(\mathrm{OCP}_1)_T$, we consider a steady state optimization problem

(SOP) : Minimize $J_s(x,u) = \dfrac{1}{2}(|Cx - z|^2 + |u|^2)$ over all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(x) + g(x)u = 0$.

We assume the following.

Assumption 1 *(SOP) has a solution* $(\bar{x},\bar{u}) = (\bar{x}(z),\bar{u}(z))$.

Also, associated with $(\mathrm{OCP}_1)_T$, we can derive a Hamilton-Jacobi equation

$$V_t(t,x) + V_x(t,x)f(x) - \frac{1}{2}V_x(t,x)g(x)g(x)^\top V_x(t,x)^\top + \frac{1}{2}|Cx - z|^2 = 0, \tag{7}$$

$$V_x(T,x) = 0, \tag{8}$$

for $V(t,x)$, where $V_t = D_tV$, $V_x = D_xV$. Defining a Hamiltonian

$$H(x,p) = p^\top f(x) - \frac{1}{2}p^\top g(x)g(x)^\top p + \frac{1}{2}|Cx - z|^2,$$

we consider the corresponding characteristic equation for (7)-(8)

$$\dot{x}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1,\dots,n \tag{9}$$

with $p_i(T) = 0, i = 1,\dots,n$. Note that since the system (5) is time-invariant, the equation corresponding to V_t is not necessary. The right side of (9) is denoted as $X_H(x,p;z)$, which is called Hamiltonian vector field associated with H. If one tries to solve (SOP) using the Lagrange multiplier method, it is necessary to solve

$$\frac{\partial H_L}{\partial x} = 0,\ \frac{\partial H_L}{\partial u} = 0,\ \frac{\partial H_L}{\partial p} = 0,$$

where $H_L(x,u,p) = p^\top(f(x) + g(x)u) + \frac{1}{2}(|u|^2 + |Cx - z|^2)$. One then immediately obtains $X_H(x,p;z) = 0$, obtaining the following fact.

Fact. A solution $(\bar{x}(z),\bar{u}(z))$ of (SOP) corresponds to an equilibrium point $(\bar{x}(z),\bar{p}(z))$ of (9) with $\bar{u}(z) = -g(\bar{x}(z))^\top\bar{p}(z)$.

Let $A_z = D_xD_pH(\bar{x}(z),\bar{p}(z)) = Df(\bar{x}(z))$, $B_z = g(\bar{x}(z))$.

Assumption 2 (A_z,B_z) *is stabilizable and* (C,A_z) *is detectable.*

Under Assumptions 1, 2, the equilibrium $(\bar{x},\bar{p})$ is a hyperbolic equilibrium for the Hamiltonian system (9) (see [26, Lemma 8]) and there exist stable and unstable manifolds for (9) at $(\bar{x},\bar{p})$ which are expressed as

$$S_z = \tilde{S} + \{(\bar{x},\bar{p})\}, \quad U_z = \tilde{U} + \{(\bar{x},\bar{p})\}. \tag{10}$$

Here, $\tilde{S}$, $\tilde{U}$ are the stable and unstable manifold of (9) in the coordinates $(\tilde{x},\tilde{p})$, where $\tilde{x} = x - \bar{x}$, $\tilde{p} = p - \bar{p}$, which is re-written as

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{p}\end{bmatrix} = \begin{bmatrix}A_z & -B_zB_z^\top\\ -C^\top C & -A_z^\top\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{p}\end{bmatrix} + o(|\tilde{x}| + |\tilde{p}|).$$

We can now state the main theorem of this subsection. Let $\pi_1 : (x,p) \mapsto x$, $\pi_2 : (x,p) \mapsto p$ be canonical projections.

Theorem 4 *Under Assumptions 1, 2, suppose that* $x_0 \in \mathrm{Int}(\pi_1(S_z))$, *where* $\mathrm{Int}(\cdot)$ *is the interior of a set in* $\mathbb{R}^n$, *and that* U_z *intersects* $p = 0$ *transversally. If* T *is taken sufficiently large, then there exists a solution* $(x_T(t,x_0),p_T(t,x_0))$ *to (9) satisfying* $x_T(0,x_0) = x_0$ *and* $p_T(T,x_0) = 0$. *If, moreover,*

$$\det D_{x_0}x_T(t,x_0) \neq 0 \text{ for } t \in [0,T], \tag{11}$$

then

$$u_T(t) := -g(x_T(t,x_0))^\top p_T(t,x_0)$$

is the local optimal solution for $(\mathrm{OCP}_1)_T$ *and turnpike inequality (6) holds for some constants* $K > 0$ *and* $\mu > 0$ *which are independent of* T.

Proof. Let $\zeta_0 = (x_0, 0)$ and $\zeta_1 = (x_1, 0)$, where $(x_1, 0) \in U_z$ and take n-dimensional discs $\{(x_0, p) \mid |p| < \rho\}$, $\{(x, 0) \mid |x - x_1| < \rho\}$, which correspond to z_0, z_1, $B(z_0, \rho) \cap \bar{D}$ and $B(z_1, \rho) \cap \bar{E}$ in Theorem 3, respectively. Then the theorem implies that for a sufficiently large $T > 0$, there exist $\rho = \rho(T) > 0$, p_0 and x_1' with $|p_0| < \rho$, $|x_1' - x_1| < \rho$ such that

$$\varphi(T, (x_0, p_0)) = (x_1', 0),$$

where $\varphi(t, (x_0, p_0))$ denotes the solution of (9) starting from (x_0, p_0). This shows that the two-point boundary value problem associated with $(\mathrm{OCP}_1)_T$ has been solved. Let $(x_T(t, x_0), p_T(t, x_0)) = \varphi(t, (x_0, p_0))$. Then, the theorem also says that there exist $K' > 0$ and $\mu > 0$ such that

$$|x_T(t, x_0) - \bar{x}| + |p_T(t, x_0) - \bar{p}| < K'[e^{-\mu t} + e^{-\mu(T-t)}] \quad \text{for } 0 \leqslant t \leqslant T.$$

Since $|u_T(t) - \bar{u}| \leqslant \sup \|g(x)\| |p(t, x_0) - \bar{p}|$, (6) holds with $K = 2K'(1 + \sup \|g(x)\|)$, where supremum is taken along the trajectory. The condition (11) guarantees that there exists a C^1 Lagrangian submanifold in a neighborhood of this trajectory whose projection to the x-space is surjective at each point of the trajectory and this implies the existence of a C^2 solution $V(t, x)$ to (7)-(8) in the neighborhood. This is proved using generating function theory for Lagrangian submanifold (see e.g. [30, page 93 and § 2 of Appendix 7] or [50]). Then, the verification theorem in Dynamic Programming (see, e.g., [2, Theorem 5-12 on page 357]) shows that the control u^* is locally optimal. ■

Remark 3 The condition (11) guarantees that the solution V to (7) exists in a neighborhood of the trajectory $(x_T(t, x_0), p_T(t, x_0))$, $0 \leqslant t \leqslant T$. The optimality of u_T is valid only in the neighborhood. When one seeks for larger domain of existence, the non-uniqueness and non-smoothness issues of solution arise, which require the notion of viscosity solutions [3,7]. We also refer to [10] for general analysis of non-unique solutions and [34,24,25] for multiple locally optimal solutions for mechanical systems.

We next show that for small x_0, z, $(\mathrm{OCP}_1)_T$ has a solution with turnpike property using perturbation theory of stable manifold. Let $A = Df(0)(= A_0)$, $B = g(0)(= B_0)$.

Assumption 3 *(A, B) is stabilizable and (C, A) is detectable.*

Fact. Under Assumption 3, there is a neighborhood of $z = 0$ in $\mathbb{R}^r$ such that (SOP) has a unique solution for z in the neighborhood and (A_z, B_z) is stabilizable and (C, A_z) is detectable..

Proof. Consider an equation $X_H(x, p; z) = 0$. From

$$DX_H(0, 0; 0) = \begin{bmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{bmatrix},$$

the stabilizability of (A, B) and the detectability of (C, A), $DX_H(0, 0; 0)$ has no eigenvalues on the imaginary axis (see [26, Lemma 8]). Therefore, by the implicit function theorem, a unique solution $\bar{x}(z)$, $\bar{p}(z)$ for the equation exists for all z in a neighborhood of $z = 0$ and $\bar{x}(z)$, $\bar{p}(z)$ are C^1 functions of z and thus, so are A_z and B_z. For $j = 1, \cdots, n$, let $\lambda_j(z)$ be the j-th eigenvalue of A_z, which is continuous in z. Define $r_j(z)$ by

$$r_j(z) = \operatorname{rank} \begin{bmatrix} C \\ \lambda_j(z) I - A_z \end{bmatrix}.$$

The detectability of (C, A) implies $r_j(0) = n$ for $j = 1, \cdots, n$. It is known that the matrix rank is lower semicontinuous in its entries (see, e.g., [29] for a proof). Therefore $r_j(z)$ is lower semicontinuous in z and hence $r_j(z) = n$ for $j = 1, \cdots, n$ in a neighborhood of $z = 0$, from which the detectability of (C, A_z) is proved. The stabilizability of (A_z, B_z) is proved in the same manner. ■

From this Fact, Assumption 3 implies that Assumptions 1, 2 hold in a neighborhood of $z = 0$.

Corollary 5 *Under Assumption 3, for sufficiently small x_0 and z and for sufficiently large T, $(\mathrm{OCP}_1)_T$ has a solution with the turnpike property.*

Proof. From the Fact above, under Assumption 3, the Hamiltonian system (9) has stable manifold S_z and unstable manifold U_z at $(\bar{x}, \bar{p})$. For $z = 0$ the linear part of the Hamiltonian system is $\mathrm{Ham} = \begin{bmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{bmatrix}$, for which we apply the eigen structure analysis in Appendix A. Apply Lemma A.1 with $R = BB^\top$, $Q = C^\top C$ and let P and L as in the Appendix. Then, the tangent spaces $T_0 S_0$, $T_0 U_0$ of S_0, U_0 at the origin can be written as

$$T_0 S_0 = \{(u, Pu) \mid u \in \mathbb{R}^n\},$$
$$T_0 U_0 = \{(Lu, (PL + I)u) \mid u \in \mathbb{R}^n\}.$$

From the expression of $T_0 S_0$, one can take x_0 sufficiently small so that there is an n-dimensional disc D_0 in S_0 that contains the origin and x_0 in its interior. From Lemma A.2, $PL + I$ is nonsingular and therefore, $T_0 U_0$ intersects $p = 0$ transversally, which implies that there is an n-dimensional disc E_0 in U_0 that intersects $p = 0$ transversally. As $z \to 0$, the Hamiltonian vector field (9) or $X_H(x, p, z)$ can be arbitrarily close to $X_H(x, p; 0)$ in the C^1 topology in an appropriate compact set. The stable manifold theory (see, e.g., [35, Theorem 6.2]) ensures that there exists a small z so that there are n-dimensional discs $D_z \subset S_z$, $E_z \subset U_z$ that are close enough to D_0, E_0, respectively, in the C^1-topology. For this z, it holds that $x_0 \in \mathrm{Int}(\pi_1(D_z))$ and E_z intersects $p = 0$ transversally. Now, all the hypotheses in Theorem 4 are satisfied. ■

Next corollary is proved in [39,48] in the study of the turnpike property for infinite dimensional systems under slightly more restrictive conditions (controllability and observability rather than stabilizability and detectability). Their proofs are based on the estimates on adjoint variables in the linear Hamiltonian system (9) which is derived as a necessary condition of optimality. Here, we give an alternative proof using the geometric picture in Theorem 4.

Corollary 6 *Suppose that the system (5) is linear, that is, $f(x) = Ax$ and $g(x) = B$ with real constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Under Assumption 3, $(\mathrm{OCP}_1)_T$ has the global solution $u^*(t)$, $0 \leqslant t \leqslant T$ for any $z \in \mathbb{R}^r$. Moreover, turnpike inequality (6) holds.*

Proof. We use some of the notations from the proof of Corollary 5. The unique solution $(\bar{x}, \bar{p})$ to (SOC) is expressed as $\left[\begin{smallmatrix}\bar{x}\\ \bar{p}\end{smallmatrix}\right] = -\mathrm{Ham}^{-1}\left[\begin{smallmatrix}0\\ C^\top z\end{smallmatrix}\right]$. U_z and S_z in (10) can be written as

$$\begin{aligned} S_z &= \{(u, Pu) \,|\, u \in \mathbb{R}^n\} + \{(\bar{x}, \bar{p})\},\\ U_z &= \{(Lu, (PL + I)u) \,|\, u \in \mathbb{R}^n\} + \{(\bar{x}, \bar{p})\}. \end{aligned}$$

It is readily seen that $x_0 \in \mathrm{Int}(\pi_1(S))$ for any $x_0 \in \mathbb{R}^n$ and U intersects $p = 0$ transversally for any $z \in \mathbb{R}^r$. The condition (11) is equivalent to the nonsingularity of (1,1)-block in $\exp[t\mathrm{Ham}]$, which is proved in Lemma A.3. ■

Remark 4 (1) Although the problem in Corollary 6 is linear, it is not an easy task to explicitly write down the solution for (7)-(8) except for $z = 0$. This corollary, however, says that the solution globally exists.
(2) As is discussed in [40,49,37], relaxing the smallness conditions in Corollary 5 is one of major challenges in the research of nonlinear turnpike. In § 4.1, we show a class of nonlinear OCPs for which turnpike occurs for all z by explicitly analyzing unstable manifold.

3.2 *The OCP with state variables specified at the terminal time*

In this subsection, we consider an OCP for (5) with arbitrarily specified terminal states. Let $x_f \in \mathbb{R}^n$ be given. Let us define cost functional

$$J_2(u) = \frac{1}{2}\int_0^T x(t)^\top C^\top C x(t) + |u|^2\, dt,$$

and consider

$(\mathrm{OCP}_2)_T$: Find a control $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $J_2(u)$ along (5) is minimized over all $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $x(T) = x_f$.

With Assumption 3, the corresponding steady state optimization problem has a unique solution $(\bar{x}, \bar{u}) = (0, 0)$ around the origin. The Hamilton-Jacobi equation associated with $(\mathrm{OCP}_2)_T$ is

$$\begin{aligned} &V_t(t, x) + V_x(t, x)f(x)\\ &\quad - \frac{1}{2}V_x(t, x)g(x)g(x)^\top V_x(t, x)^\top + \frac{1}{2}x^\top C^\top C x = 0. \end{aligned} \tag{12}$$

The Hamiltonian in this case is

$$H(x, p) = p^\top f(x) - \frac{1}{2}p^\top g(x)g(x)^\top p + \frac{1}{2}x^\top C^\top C x,$$

and the corresponding characteristic equation for (12) is

$$\dot{x}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \dots, n \tag{13}$$

with $x(0) = x_0$ and $x(T) = x_f$.

Under Assumptions 3, the Hamiltonian system (13) can be written as

$$\frac{d}{dt}\begin{bmatrix} x\\ p\end{bmatrix} = \mathrm{Ham}\begin{bmatrix} x\\ p\end{bmatrix} + o(|x| + |p|),$$

and the origin is a hyperbolic equilibrium with n stable and n unstable eigenvalues. Let S and U be the stable and unstable manifolds of (13) at the origin.

Theorem 7 *Under Assumption 3, suppose that $x_0 \in \mathrm{Int}(\pi_1(S))$ and $x_f \in \mathrm{Int}(\pi_1(U))$. If $T > 0$ is taken sufficiently large, there exists a solution $(x_T(t, x_0), p_T(t, x_0))$ to (13) satisfying $x(0) = x_0$ and $x(T) = x_f$. If, moreover,*

$$\det D_{x_0} x(t, x_0) \neq 0 \text{ for } t \in [0, T], \tag{14}$$

then

$$u_T(t) = -g(x_T(t, x_0))^\top p_T(t, x_0)$$

is the local optimal solution for $(\mathrm{OCP}_2)_T$ and turnpike inequality (6) hols for some $K > 0$, $\mu > 0$ independent of T.

Proof. Let $\zeta_0 = (x_0, 0)$, $\zeta_1 = (x_f, 0)$, $\{(x_0, p) \,|\, |p| < \rho\}$ and $\{(x_f, p) \,|\, |p| < \rho\}$ which correspond to z_0, z_1,

$B(z_0,\rho)\cap\bar{D}$ and $B(z_1,\rho)\cap\bar{E}$ in Theorem 3, respectively. Then, for $T>0$ sufficiently large, there exist $\rho>0$, p_0 and p_1 with $|p_0|,|p_1|<\rho$ such that a solution to (13) connecting (x_0,p_0) and (x_f,p_1) exists. The rest of the proof is almost the same as Theorem 4. ■

Corollary 8 *Let us additionally impose the controllability of (A,B) in Assumption 3. Then, for sufficiently small $|x_0|$ and $|x_f|$ and sufficiently large T, the local optimal control exists and turnpike inequality (6) holds.*

Proof. We again employ the eigenstructure analysis (A.2). The tangent spaces of S and U at the origin are written as

$$T_0S=\{(u,Pu)\,|\,u\in\mathbb{R}^n\},$$
$$T_0U=\{(u,(PL+I)L^{-1}u)\,|\,u\in\mathbb{R}^n\}.$$

The latter is obtained by showing, using the controllability of (A,B), that L is strictly negative definite (Lemma A.2). Therefore, $x_0\in\mathrm{Int}(\pi_1(S))$ and $x_f\in\mathrm{Int}(\pi_1(U))$ for sufficiently small $|x_0|,|x_f|$. It is seen that the condition (14) holds for these $|x_0|$, $|x_f|$ (making them smaller if necessary) from the analysis on $\Phi_{11}(t)$ in the proof of Theorem 4. ■

Remark 5 (i) The linear counterpart of Corollary 8 is in [52, Lemma 5 on page 383] where the authors use anti-stabilizing solution P_u for the Riccati equation. In this case, the turnpike holds for all x_0 and x_f. It can be shown that $P_u=(PL+I)L^{-1}$. Note that in Corollary 8 we only need the detectability condition. Corollary 8 is obtained in [1, Properties 4.1 and 4.2] using Hamilton-Jacobi theory under unusual nonlinear controllability and observability conditions. Compared with the conditions, we use only the *linear* controllability and detectability which can be easily checked. The authors of [49, Theorem 1 on page 87] obtain similar results to Corollary 8 with more general terminal conditions.

(ii) Corollary 8 states that the turnpike occurs for small initial and terminal states under linear stabilizability and detectability. Relaxing the smallness conditions is one of major challenges in (OCP_2). In § 4.2, we will give a class of nonlinear systems for which the turnpike occurs for all initial states. This is done with the aid of the result in [44] (see Proposition B.1) giving an estimate on the region for stable manifold in terms of nonlinear stabilizability. In the example in § 4.2, the unstable manifold is linear and a geometric condition in Theorem 7 is readily verified.

4 Examples

4.1 *Problem* (OCP_1)

In this subsection, we show a class of nonlinear systems where the turnpike occurs in (OCP_1) for all targets z. Let us consider the following class of nonlinear control systems

$$\begin{cases}\dot{x}_1=A_1x_1+A_2(x_1,x_2)x_1\\ \dot{x}_2=A_3x_2+B_2u,\end{cases}\tag{15}$$

where A_1 is an $n_1\times n_1$ Hurwitz matrix, $A_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_1\times n_1}$ is a C^2 function and $u\in\mathbb{R}^m$ is the control input. Assume that (A_3,B_2) is stabilizable and $A_2(0,x_2)=0$ for all $x_2\in\mathbb{R}^{n_2}$. The cost function is

$$J_1=\frac{1}{2}\int_0^T|u|^2+|C_1x_1-z_1|^2+|C_2x_2-z_2|^2\,dt,\tag{16}$$

where C_1, C_2 are constant matrices with appropriate dimensions, (C_2,A_3) is detectable and $z_1\in\mathbb{R}^{r_1}$, $z_2\in\mathbb{R}^{r_2}$ are given constant vectors.

The corresponding Hamiltonian system for this problem is

$$\begin{cases}\dot{x}_1=A_1x_1+A_2(x_1,x_2)x_1\\ \dot{x}_2=A_3x_2-B_2B_2^\top p_2\\ \dot{p}_1=-C_1^\top(C_1x_1-z_1)-A_1^\top p_1\\ \qquad -D_{x_1}\left[p_1^\top A_2(x_1,x_2)x_1\right]^\top\\ \dot{p}_2=-C_2^\top(C_2x_2-z_2)\\ \qquad -A_3^\top p_2-D_{x_2}\left[p_1^\top A_2(x_1,x_2)x_1\right]^\top.\end{cases}\tag{17}$$

Using the stabilizability and detctability of (A_3,B_2) and (C_3,A_3), it can be seen that there is an equilibrium $(0,x_{20}(z_2),p_{10}(z_1),p_{20}(z_2))$ for (17). At this equilibrium, the linearized matrix is

$$\begin{bmatrix}A_1 & 0 & 0 & 0\\ 0 & A_3 & 0 & -B_2B_2^\top\\ -C_1^\top C_1-\Gamma & 0 & -A_1^\top & 0\\ 0 & -C_2^\top C_2 & 0 & -A_3^\top\end{bmatrix},$$

where $\Gamma=\Gamma(z_1,z_2)$ is an $n_1\times n_1$ symmetric matrix and therefore, it can be seen that it is a hyperbolic equilibrium.

Let P_1, P_3 and S_3 be solutions for

$$P_1A_1+A_1^\top+C_1^\top C_1+\Gamma=0,$$
$$P_3A_3+P_3A_3^\top-P_3B_2B_2^\top P_3+C_2^\top C_2=0,$$
$$(A_3-B_2B_2^\top P_3)S_3+S_3(A_3-B_2B_2^\top P_3)^\top=B_2B_2^\top,$$

with $P_1 = P_1^\top$, $P_3 \geqslant 0$, $S_3 \geqslant 0$ and $A_3 - B_2 B_2^\top P_3$ being Hurwitz. Using a linear coordinate transformation (see Appendix A)

$$\begin{bmatrix} x_1 \\ x_2 - x_{20} \\ p_1 - p_{10} \\ p_2 - p_{20} \end{bmatrix} = T \begin{bmatrix} p_1' \\ p_2' \\ x_1' \\ x_2' \end{bmatrix}; \quad T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & S_3 \\ P_1 & 0 & I & 0 \\ 0 & P_3 & 0 & I + P_3 S_3 \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I + S_3 P_3 & 0 & -S_3 \\ -P_1 & 0 & I & 0 \\ 0 & -P_3 & 0 & I \end{bmatrix},$$

the Hamiltonian system (17) is rewritten as

$$\begin{cases} \dot{p}_1' = A_1 p_1' + \psi_1(p_1', p_2', x_1', x_2') \\ \dot{p}_2' = (A_3 - B_2 B_2^\top P_3) p_2' + \psi_2(p_1', p_2', x_1', x_2') \\ \dot{x}_1' = -A_1^\top x_1' + \psi_3(p_1', p_2', x_1', x_2') \\ \dot{x}_2' = -(A_3 - B_2 B_2^\top P_3)^\top x_2' + \psi_4(p_1', p_2', x_1', x_2'), \end{cases}$$

where ψ_j, $j = 1, \ldots, 4$, are appropriately computed higher order terms. Since $\psi_j(0, 0, x_1', x_2')$=0, $j = 1, \ldots, 4$, for all x_1', x_2', the unstable manifold U at the equilibrium is the affine space $p_1' = p_2' = 0$, or

$$U = \{x_1 = 0, (I + S_3 P_3)(x_2 - x_{20}) - S_3(p_2 - p_{20}) = 0\}.$$

Since $I + S_3 P_3$ is nonsingular, which is shown using Lemma A.2 and Sylvester's determinant identity, for any z_1, z_2, U intersects $p_1 = p_2 = 0$ transversally. Now, using Theorem 4, for any z_1, z_2, if the initial point $(x_1(0), x_2(0))$ is close enough to $(0, x_{20})$, the optimal control for (15)-(16) possesses the turnpike property.

As an example of the class of systems, a turnpike trajectory for a nonlinear optimal control problem

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \quad \dot{x}_2 = u \tag{18a}$$

$$J_1 = \frac{1}{2} \int_0^T u^2 + (x_1 - z_1)^2 + (x_2 - z_2)^2 \, dt \tag{18b}$$

is depicted in Fig. 2, where a solution of (SOP) is $(0, z_2, -z_1, 0)$.

4.2 *Problem* (OCP_2)

Next, we show a class of nonlinear control systems for which estimates on (un)stable manifold of Hamiltonian systems obtained in [44] are effective for the prediction of turnpike.

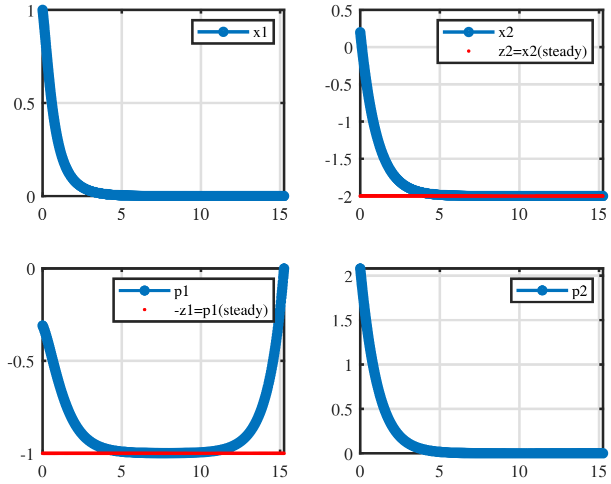

Fig. 2. Optimal trajectory for (18) $(x_1(0), x_2(0)) = (1, 0.2)$ and $(z_1, z_2) = (1, -2)$.

Let us consider an $(n_1 + n_2)$-dimensional system represented in Byrnes-Isidori normal form [5] for globally exponentially minimum phase nonlinear systems

$$\begin{cases} \dot{x} = q(x, y_1) \\ \dot{y}_1 = y_2 \\ \cdots \\ \dot{y}_{n_2} = u, \end{cases} \tag{19}$$

where $x \in \mathbb{R}^{n_1}$ and $q : \mathbb{R}^{n_1+1} \to \mathbb{R}^{n_1}$ is a smooth map with $q(0, 0) = 0$. We assume that $\dot{x} = q(x, 0)$ is globally exponentially stable. It is known that (19) is globally exponentially stabilizable via a smooth feedback. Therefore, representing $y = (y_1, \ldots, y_{n_2})$, for a cost functional

$$L(u, x, y) = \frac{1}{2}(|u|^2 + |C_1 x|^2 + |C_2 y|^2),$$

the associated Hamiltonian system is hyperbolic at the origin if C_2 and the matrix defining y-dynamics is a detectable pair. If, in addition, $|C_1 x|^2$ and $q(x, y_1)$ satisfy the growth condition in Proposition B.1-(iv) with respect to x, the stable manifold S of the Hamiltonian system satisfies $\pi_1(S) = \mathbb{R}^{n_1+n_2}$. Therefore, from Corollary 8, the OCP has a solution for all x_0 and for sufficiently small x_f that exhibits turnpike if the linear detectability condition at the origin for $h = |C_1 x|^2 + |C_2 y|^2$ is satisfied and T is taken large enough.

As a numerical example, consider (18a), which is in Byrnes-Isidori normal form (see e.g., [5]), with

$$J_2 = \frac{1}{2} \int_0^T u^2 + x_1^2 + x_2^2 \, dt. \tag{20}$$

Introducing a cut-off function on x_2, the result in [44] is applied to confirm that the turnpike occurs for all initial condition $x_0 = (x_1(0), x_2(0))$ and terminal states $x_f \in \pi_1(U)$, where U is the unstable manifold of the Hamiltonian system at the origin. Similarly to the previous subsection, U is described as

$$U = \{x_1 = 0,\ x_2 + p_2 = 0\}.$$

Figs. 3, 4 show the turnpike trajectory of the optimal control problem (18a)-(20) with $x_0 = (12, 12)$, $x_f = (0, 5)$. In Fig. 4, $x_1(t)$, $x_2(t)$, $p_1(t)$, $p_2(t)$ are depicted for $t \in [0, 0.1]$ while the last figure shows $p_2(t)$ for $t \in [0.1, 10]$. From these figures, one sees that starting from $x_0 = (12, 12)$ at $t = 0$, the states and costates rapidly grow during the time span $[0, 0.02]$ and go to the steady state optimal solution (the origin) by the time $t = 0.1$ and then, the states reach the destination $x_f = (0, 5)$ at $t = 10$. The peak of this growth increases as $|x_0|$ increases. This growth of states is called "peaking phenomenon" of nonlinear stabilization [47] and it is interesting to see that peaking phenomenon appears in turnpike trajectory.

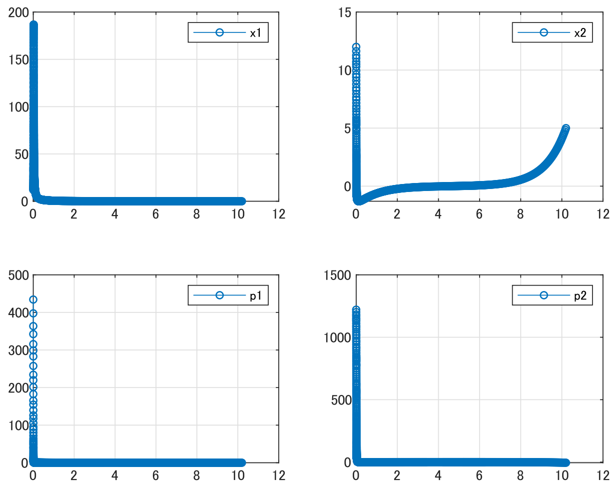

Fig. 3. Optimal trajectory for (18a)-(20) with $x_0 = (12, 12)$ and $x_f = (0, 5)$.

5 Discussions

The geometric approach proposed in the present paper may be applied to more general cases where turnpike phenomena need more sophisticated analyses. Here, we discuss two kinds of extensions.

5.1 *Global analysis when (SOP) admits multiple solutions*

When (SOP) admits multiple solutions, multiple equilibria appear in associated Hamiltonian systems. If they are all hyperbolic, the λ-lemma still applies to draw pictures of flows around stable and unstable manifolds that are separatrices dividing the phase space (see, e.g., [35, p.87 Corollary 1]).

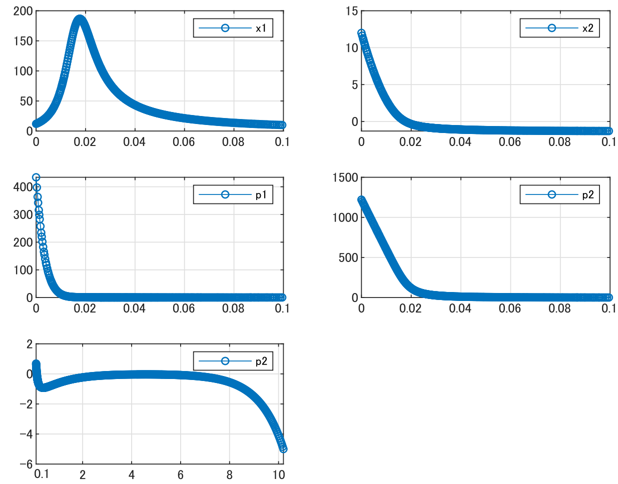

Fig. 4. Enhanced figures of Fig. 3 for time spans $[0, 0.1]$ and $[0.1, 10]$ (last figure).

Let us consider $(\mathrm{OCP}_2)_T$ for

$$\dot{x} = -x + x^2 + u, \tag{21a}$$

$$J_2(u) = \frac{1}{2}\int_0^T u^2\, dt. \tag{21b}$$

The associated Hamiltonian system has three equilibrium points; $(0, 0)$, $(1, 0)$ and $(1/2, -1/4)$, the first two of which are the global solution of (SOP) and hyperbolic. Fig. 5 shows stable and unstable manifolds, closed orbits around $(1/2, -1/4)$ and heteroclinic orbits connecting $(0, 0)$ and $(0, 1)$. From this figure and

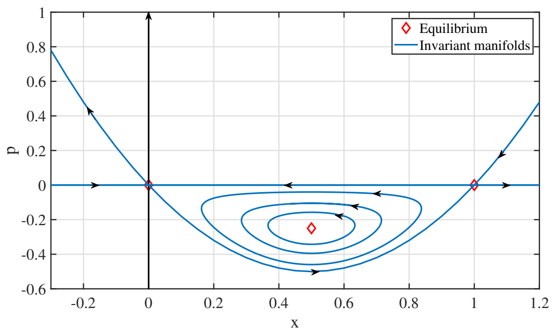

Fig. 5. Invariant manifolds of Hamiltonian system for (21).

using the geometric method in the present paper, one immediately sees that for any initial point $x(0)$ and final point x_f, solution for $(\mathrm{OCP}_2)_T$ with large T exists. For instance, trajectory in x-p space, corresponding optimal input and optimal trajectory are depicted in Fig. 6 for $x(0) = 1.5$, $x_f = -1$ (note that $u = -p$). Although the input response looks like turnpike, the response of x for $u \sim 0$ is not stationary but steady motion with nonzero velocity. Fig. 7 shows optimal trajectory and control response for $x(0) = -0.1$, $x_f = 1.4$. Nonzero control is necessary to drive x against stable

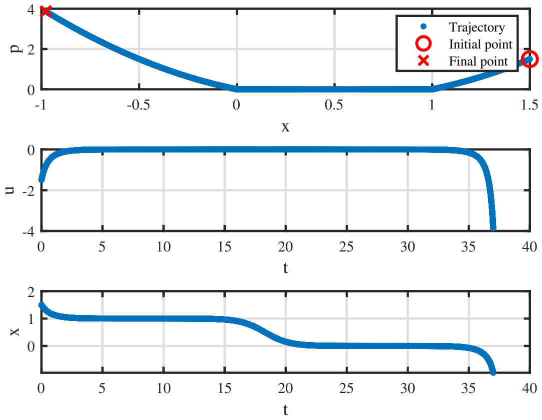

Fig. 6. Prolonged turnpike due to two equilibrium points.

vector field. The ratio of the time duration for nonzero control for the overall horizon can be arbitrarily small as $T \to \infty$ and in this sense, this can be also considered turnpike phenomenon.

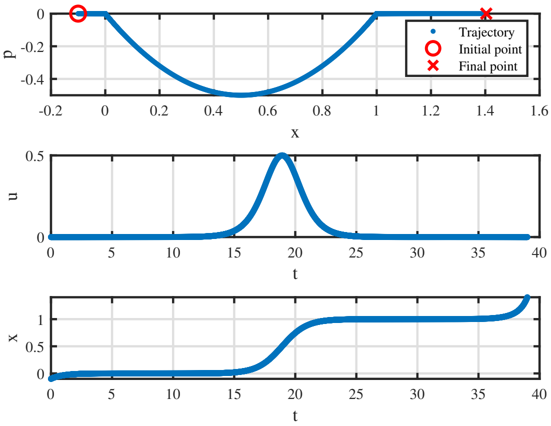

Fig. 7. Bump turnpike due to two equilibrium points.

As for (OCP_1), when multiple global minimizers exist, an interesting question is raised in [36] as to which minimizer attracts turnpike for wider initial conditions. It is interesting to study how the geometry of these invariant manifolds affects turnpike occurrence and its strength in terms of the question.

5.2 *Non-hyperbolic Hamiltonian systems*

In [13], a concept of velocity turnpike or time-varying turnpike arising in mechanical systems is proposed combining trim primitives and turnpike properties. Motivated by that, the authors in [38] consider turnpike properties when detectability (ovservability) is not satisfied. A common feature in these cases is that associated Hamiltonian systems have zero eigenvalues. It is then interesting to consider the application of the λ-lemma for normally hyperbolic invariant manifolds [8] combining the classification result on Hamiltonian and symplectic matrices [28].

6 Conclusions

In this paper, using techniques from dynamical system theory such as invariant manifolds and the λ-lemma, we showed that turnpike-like behavior naturally appears in hyperbolic dynamical systems. This is then applied to analyze Hamiltonian systems describing controlled trajectories to obtain sufficient conditions for optimal controls yielding the turnpike to exist.

The framework proposed in the paper provides geometric insights to understand the turnpike. More specifically, it makes possible to interlink the turnpike analysis and the nonlinear stability and/or stabilizability, for which an enormous amount of research effort has been devoted for several decades. As examples, in § 4, we showed classes of nonlinear systems for which target z for (OCP_1) and initial states can be taken arbitrarily large for (OCP_2).

Since our interests were to discover geometric nature in turnpike, we focused on OCPs without constraints and exponential turnpike. Future works include applications of this approach to more specific problems and considering OCPs with constraints, for which we mention an attempt to analyze the turnpike in the maximum hands-off control [45].

Acknowledgement. The authors would like to thank Emmanuel Trélat and Lars Grüne for their comments on the early version of the manuscript. The authors are also grateful to Dario Pighin for valuable discussions.

Appendix

A Results related with Riccati equations and linear Hamiltonian systems

Let us consider Riccati equation

$$PA + A^\top P - PRP + Q = 0, \tag{A.1}$$

where $A, R, Q \in \mathbb{R}^{n\times n}$ are constant matrices with $R, Q \geqslant 0$. Suppose that (A, R) is stabilizable and (Q, A) is detectable. The following are known.

Lemma A.1 *(i) All eigenvalues λ of $\mathrm{Ham} = \begin{bmatrix} A & -R \\ -Q & -A^\top \end{bmatrix}$ satisfy $\mathrm{Re}\,\lambda \neq 0$ (see [26, Lemma 7]).*

(ii) *There is a solution $P \geqslant 0$ to (A.1) such that $A_c := A - RP$ is Hurwitz (see [16, Corollary 1 on page 92], [26, Theorem 1]).*

(iii) *Let $L \leqslant 0$ be a solution to a Lyapunov equation*

$$LA_c^\top + A_cL = R,$$

then $\left[\begin{smallmatrix} I & L \\ P & PL+I \end{smallmatrix}\right]$ is a symplectic matrix and its inverse is $\left[\begin{smallmatrix} LP+I & -L \\ -P & I \end{smallmatrix}\right]$ (see [31, Lemma 2.5], [43, page 1933]).

(iv) *The Hamiltonian matrix* Ham *is block-diagonalized as*

$$\mathrm{Ham}\begin{bmatrix} I & L \\ P & PL+I \end{bmatrix} = \begin{bmatrix} I & L \\ P & PL+I \end{bmatrix}\begin{bmatrix} A_c & 0 \\ 0 & -A_c{}^\top \end{bmatrix} \tag{A.2}$$

(see [31, Lemma 2.5], [43, page 1933]).

The following lemma can be considered as a dual version of Theorem 2 in [16, page 90], for which simplified proofs are given for the sake of self-containedness.

Lemma A.2 *$PL+I$ is nonsingular. If, in addition, (A, R) is controllable, then $L < 0$ (negative definite).*

Proof. Let $V := PL + I$. From (A.2) we have

$$AL - RV = -LA_c^\top \tag{A.3a}$$
$$-QL - A^\top V = -VA_c^\top \tag{A.3b}$$

We show that the condition $\dim \mathrm{Ker}\, V \geqslant 1$ leads to a contradiction. It can be shown from (A.3) that $0 \neq v \in \ker V$ satisfies $QLv = 0$, $VA_c^\top v = 0$ using $LV = V^\top L$ and $Q \geqslant 0$, showing that $\mathrm{Ker}\, V$ is $A_c^\top$-invariant. Thus, we may assume that v is an eigenvector of $A_c^\top$ with eigenvalue λ with $\mathrm{Re}\,\lambda < 0$. From (A.3b), we have $ALv = -A_c^\top v = -\lambda Lv$ and therefore $(-\lambda I - A)Lv = 0$. This shows that $\left[\begin{smallmatrix} Q \\ -\lambda I - A \end{smallmatrix}\right] Lv = 0$. With $\mathrm{Re}\,(-\lambda) > 0$, the detectability of (Q, A) implies $Lv = 0$. This shows that $\left[\begin{smallmatrix} L \\ PL+I \end{smallmatrix}\right] v = 0$ with $v \neq 0$, which contradicts Lemma A.1(iii). The second statement can also be proved in a similar way, deriving $\lambda u = A^\top u$, $Ru = 0$ for $0 \neq u \in \mathrm{Ker}\, L$ and a contradiction. ∎

Lemma A.3 *Let*

$$\begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix} = \exp[t\mathrm{Ham}],$$

where $\Phi_{ij}(t)$, $i, j = 1, 2$, are $n \times n$ matrix functions of t. When (A, R) is stabilizable and (Q, A) is detectable, $\Phi_{11}(t)$ is nonsingular for $t \geqslant 0$.

Proof. Using (A.2),

$$\begin{aligned}\Phi_{11}(t) &= \exp[tA_c] \\ &\quad \times \left\{I + \left(L - \exp[-tA_c]L\exp[-tA_c^\top]\right)P\right\} \\ &= \exp[tA_c](I + \tilde{L}(t)P),\end{aligned}$$

where we have set $\tilde{L} := L - \exp[-tA_c]L\exp[-tA_c^\top]$. Since $\tilde{L}(0) = 0$ and

$$\begin{aligned}\frac{d}{dt}\tilde{L}(t) &= \exp[-tA_c](A_cL + LA_c^\top)\exp[-tA_c^\top] \\ &= \exp[-tA_c]R\exp[-tA_c^\top] \geqslant 0\end{aligned}$$

by Lemma A.1(iii), $\tilde{L}(t) \geqslant 0$ for $t \geqslant 0$. If $\Phi_{11}(t)\eta = 0$ for some $t \geqslant 0$ and $\eta \in \mathbb{C}^n$, then we have $(I + \tilde{L}(t)P)\eta = 0$ and therefore $\eta^*P\eta + \eta^*P\tilde{L}(t)P\eta = 0$. This implies $P\eta = 0$, $\tilde{L}(t)P\eta = 0$ and we have $\eta = 0$. ∎

B Existence of infinite horizon optimal control and stable manifold of Hamiltonian systems

This appendix introduces a result in [44] on the existence of infinite horizon optimal control. The main result in the paper is under simpler growth conditions than those given below, but is more restrictive to apply.

Let $U \subset \mathbb{R}^n$ be an open set containing the origin. A nonlinear system (5) is said to be *C^1-exponentially stabilizable in U* if there exists a C^1 feedback control $u = k(x)$ with $k(0) = 0$ such that the the closed loop system is exponentially stable with respect to U. Let $h(x)$ be a C^2 nonnegative function defined in $\mathbb{R}^n$ satisfying $h(0) = 0$, $Dh(0) = 0$.

For system (5), let $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ and rewrite it as

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= f(x_1, x_2) + g(x_1, x_2)u \\ &= \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} u,\end{aligned}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$, $g_j : \mathbb{R}^n \to \mathbb{R}^{n_j \times m}$, $j = 1, 2$. Let $\varphi_R : \mathbb{R}^{n_2} \to \mathbb{R}$ be a C^∞ cutoff function such that $\varphi_R(x_2) = 1$ for $|x_2| < R$ and $\varphi_R(x_2) = 0$ for $|x_2| \geqslant R + 1$. Define $\tilde{f}_R(x_1, x_2) := f(x_1, \varphi_R(x_2)x_2)$ and $\tilde{g}_R(x_1, x_2) := g(x_1, \varphi_R(x_2)x_2)$.

Assumption B.1 *(i) System (5) is C^1-exponentially stabilizable in Ω, where Ω is an open set in $\mathbb{R}^n$ containing the origin.*

(ii) *For a nonnegative C^1 function $h(x)$, there exist positive constants p, ρ, c_h such that $h(x) \geqslant c_h|x|^p$ for $|x| > \rho$.*

(iii) *The pair of linearizations of f and h at the origin is detectable.*

(iv) *For any $R > 0$, there exist constants $c_f > 0$, $c_g > 0$, $0 \leqslant \theta < 1$, which may depend on R, such that*

$$|\tilde{f}_R(x)| \leqslant c_f|x|^{p+\theta},$$
$$\|\tilde{g}_R(x)\| \leqslant c_g|x|^{p/2+\theta},$$

for sufficiently large $x \in \mathbb{R}^n$.

(v) *There exist constants $c_{f2} > 0$, $c_{g2} > 0$ and $0 \leqslant \theta_2 < 1$ such that*

$$|f_2(x_1, x_2)| \leqslant c_{f2}|x_2|^{p+\theta_2},$$
$$\|g_2(x_1, x_2)\| \leqslant c_{g2}|x_2|^{p/2+\theta_2},$$

for all $x_1 \in \mathbb{R}^{n_1}$ and sufficiently large $x_2 \in \mathbb{R}^{n_2}$.

Proposition B.1 *Under Assumption B.1, for OCP (5) and*

$$J = \int_0^\infty |u(t)|^2 + h(x(t))\, dt, \tag{B.1}$$

there exists an optimal control for $x(0) \in \Omega$. Furthermore, for a Hamiltonian system associated with OCP (5)-(B.1), a stable manifold S at the origin exists with the projection property $\Omega \subset \pi_1(S)$.

References

[1] Brian D. O. Anderson and Peter V. Kokotovic. Optimal control problems over large time intervals. *Automatica*, 23(3):355–363, 1987.

[2] Michael Athans and Peter L. Falb. *Optimal Control: An Introduction to the Theory and Its Applications*. McGrow-Hill, New York, 1966.

[3] Martino Bardi and Itaro Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, 1997.

[4] Julian Berberich, Johannes Köhler, Frank Allgöwer, and Mathias A Müller. Indefinite linear quadratic optimal control: Strict dissipativity and turnpike properties. *IEEE Control Systems Letters*, 2(3):399–404, 2018.

[5] Christopher I. Byrnes and Albelto Isidori. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. Automat. Control*, 36(10):1122–1137, 1991.

[6] Dean A. Carlson, Alain B. Haurie, and Arie Leizarowitz. *Infinite Horizon Optimal Control*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 1991.

[7] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[8] Jacky Cresson and Stephen Wiggins. A λ-lemma for normally hyperbolic invariant manifolds. *Regular and Chaotic Dynamics*, 20(1):94–108, 2015.

[9] Tobias Damm, Lars Grüne, Marleen Stieler, and Karl Worthmann. An exponential turnpike theorem for dissipative discrete time optimal control problems. *SIAM J. Control Optim.*, 52(3):1935–1957, 2014.

[10] Martin V. Day. On Lagrange manifolds and viscosity solutions. *Journal of Mathematical Systems, Estimation and Control*, 18:369–372, 1998. `http://www.math.vt.edu/people/day/research/LMVS.pdf`.

[11] Robert Dorfman, Paul A. Samuelson, and Robert M. Solow. *Linear Programming and Economic Analysis*. McGraw-Hill, New York, 1958.

[12] Timm Faulwasser and Dominique Bonvin. Exact turnpike properties and economic nmpc. *European Journal of Control*, 35:34–41, 2017.

[13] Timm Faulwasser, Kathrin Flaßamp, Sina Ober-Blöbaum, and Karl Worthmann. Towards velocity turnpikes in optimal control of mechanical systems. *IFAC-PapersOnLine*, 52(16):490–495, 2019. In Proc. *11th IFAC Symposium on Nonlinear Control Systems, NOLCOS 2019*.

[14] Timm Faulwasser, Milan Korda, Colin N. Jones, and Dominique Bonvin. On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81:297–304, 2017.

[15] Timm Faulwasser and Alexander. Murray. Turnpike properties in discrete-time mixed-integer optimal control. *IEEE Control Systems Letters*, 4(3):704–709, 2020.

[16] Bruce A. Francis. *A Course in H_∞ Control Theory*. Springer, 1986.

[17] Lars Grüne. Economic receding horizon control without terminal constraints. *Automatica*, 49(3):725–734, 2013.

[18] Lars Grüne and Roberto Guglielmi. Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems. *SIAM J. Control Optim.*, 56(2):1282–1302, 2018.

[19] Lars Grüne and Matthias A. Müller. On the relation between strict dissipativity and turnpike properties. *Syst. Control Lett.*, 90:45–53, 2016.

[20] Lars Grüne, Manuel Schaller, and Anton Schiela. Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations. *Journal of Differential Equations*, 268(12):7311–7341, 2020.

[21] Martin Gugat and Falk M. Hante. On the turnpike phenomenon for optimal boundary control problems with hyperbolic systems. *SIAM Journal on Control and Optimization*, 57(1):264–289, 2019.

[22] Martin Gugat, Emmanuel Trélat, and Enrique Zuazua. Optimal Neumann control for the 1d wave equation: Finite horizon, infinite horizon, boundary tracking terms and the turnpike property. *Systems and Control Letters*, 90:61–70, 2016.

[23] Jack K. Hale. *Ordinary Differential Equations*. Krieger Publishing Company, Florida, 2nd edition, 1980.

[24] Takamasa Horibe and Noboru Sakamoto. Optimal swing up and stabilization control for inverted pendulum via stable manifold method. *IEEE Trans. on Control System Technology*, 26(2):708–715, 2017.

[25] Takamasa Horibe and Noboru Sakamoto. Nonlinear optimal control for swing up and stabilization of the Acrobot via stable manifold approach: Theory and experiment. *IEEE Trans. on Control System Technology*, 27(6):2374–2387, 2019.

[26] Vladimir Kučera. A contribution to matrix quadratic equations. *IEEE Trans. Automat. Control*, 17(3):344–347, 1972.

[27] Gontran Lance, Emmanuel Trélat, and Enrique Zuazua. Shape turnpike for linear parabolic PDE models. *Syst. Control Lett.*, 142:104733, 2020.

[28] Alan J. Laub and Kenneth Meyer. Canonical forms for symplectic and Hamiltonian matrices. *Celestial Mechanics*, 9(2):213–238, 1974.

[29] Andrew D. Lewis. Semicontinuity of rank and nullity and some consequences. https://mast.queensu.ca/~andrew/notes/pdf/2009a.pdf, 2009.

[30] Paulette Libermann and Charles-Michel Marle. *Symplectic Geometry and Analytical Mechanics*. D. Reidel, 1987.

[31] D. L. Lukes. Optimal regulation of nonlinear dynamical systems. *SIAM J. Control Optim.*, 7(1):75–100, 1969.

[32] Lionel McKenzie. Turnpike theorems for a generalized Leontief model. *Econometrica*, 31(1-2):165–180, 1963.

[33] Masaaki Nagahara, Daniel E. Quevedo, and Dragan Nešić. Maximum hands-off control: A paradigm of control effort minimization. *IEEE Trans. Automat. Control*, 61(3):735–747, 2016.

[34] Hinke M. Osinga and John Hauser. The geometry of the solution set of nonlinear optimal control problems. *Journal of Dynamics and Differential Equations*, 18(4):881–900, 2006.

[35] Jacob Palis, Jr. and Welington de Melo. *Geometric Theory of Dynamical Systems: An Introduction*. Springer-Verlag, New York, 1982.

[36] Dario Pighin. Nonuniqueness of minimizers for semilinear optimal control problems. arXiv preprint, 2020. arXiv:2002.04485.

[37] Dario Pighin. The turnpike property in semilinear control. arXiv preprint, 2020. arXiv:2004.03269.

[38] Dario Pighin and Noboru Sakamoto. The turnpike with lack of observability. arXiv preprint, 2020. arXiv:2007.14081.

[39] Alessio Porretta and Enrique Zuazua. Long time versus steady state optimal control. *SIAM J. Control Optim.*, 51(6), 2013.

[40] Alessio Porretta and Enrique Zuazua. Remarks on long time versus steady state optimal control. *Springer INdAM Series*, 15:67–89, 2016.

[41] Alain Rapaport and Pierre Cartigny. Turnpike theorems by a value function approach. *ESAIM: Control, Optimisation and Calculus of Variations*, 10(1):123–141, 2004.

[42] R. T. Rockafellar. Saddle points of Hamiltonian systems in convex problems of Lagrange. *Journal of Optimization Theory and Applications*, 12(4):367–390, 1973.

[43] Noboru Sakamoto. Analysis of the Hamilton-Jacobi equation in nonlinear control theory by symplectic geometry. *SIAM J. Control Optim.*, 40(6):1924–1937, 2002.

[44] Noboru Sakamoto. When does stabilizability imply the existence of infinite horizon optimal control in nonlinear systems? arXiv preprint, 2020. arXiv:2008.13387.

[45] Noboru Sakamoto and Masaaki Nagahara. The turnpike property in the maximum hands-off control. In *Proc. of 59th IEEE Conference on Decision and Control*, 2020.

[46] Noboru Sakamoto, Dario Pighin, and Enrique Zuazua. The turnpike property in nonlinear optimal control — A geometric approach. In *Proc. of 58th IEEE Conference on Decision and Control*, pages 2422–2427, 2019.

[47] H. J. Sussmann and P. V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. Automat. Control*, 36(4):424– 440, 1991.

[48] Emmanuel Trélat, Can Zhang, and Enrique Zuazua. Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces. *SIAM J. Control Optim.*, 56(2):1222–1252, 2018.

[49] Emmanuel Trélat and Enrique Zuazua. The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations*, 258(1):81–114, 2015.

[50] A. J. van der Schaft. L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Trans. Automat. Control*, 37(6):770–784, 1992.

[51] Lan Wen. *Differentiable Dynamical Systems: An Introduction to Structural Stability and Hyperbolicity*. American Mathematical Society, 2016.

[52] R. R. Wilde and P. V. Kokotovic. A dichotomy in linear control theory. *IEEE Trans. Automat. Control*, 17(3):382–383, 1972.

[53] Jan C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Automat. Control*, 16:621–634, 1971.

[54] Jan C. Willems. Dissipative dynamical systems-Part I, II. *Arch. Rational Mechanics and Analysis*, 45:321–393, 1972.

[55] Alexander J. Zaslavski. *Turnpike properties in the calculus of variations and optimal control*. Springer, 2006.

[56] Enrique Zuazua. Large time control and turnpike properties for wave equations. *Annual Reviews in Control*, 44:199–210, 2017.