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Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks

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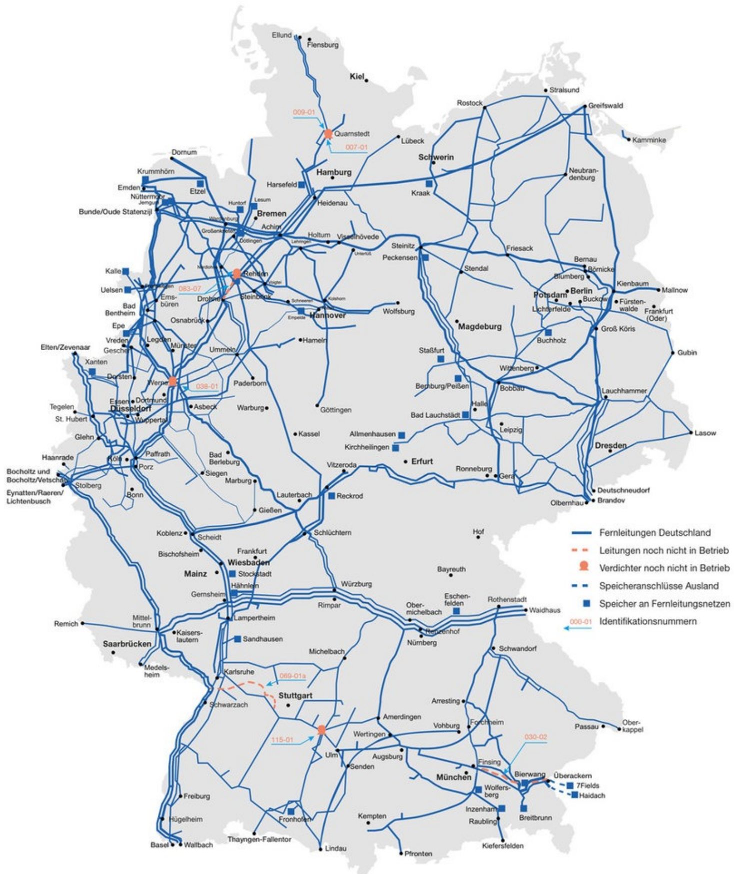
FAU Erlangen-Nürnberg
July 27, 2021

CONTINUOUS OPTIMIZATION

Motivation



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Motivation

Consider the optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}(\ g(x,\xi) \leq 0\ ) \ \geq \ \alpha \end{aligned}$$

with objective function f, constraint g, decision vector x, random variable ξ (with probability distribution and density function) and probability level α.

We have:

$$\mathbb{P}(g(x,\xi) \leq 0) = \int_{M(x)} \varrho_\xi(z)\, dz,$$

with

$$M(x) = \{\omega \in \Omega \mid g(x,\xi(\omega)) \leq 0\}.$$

Motivation

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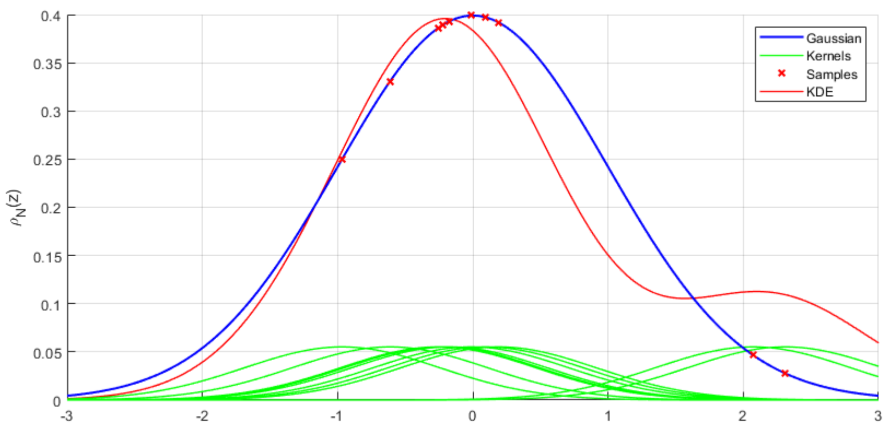
Is there a „better“ way to compute this probability?

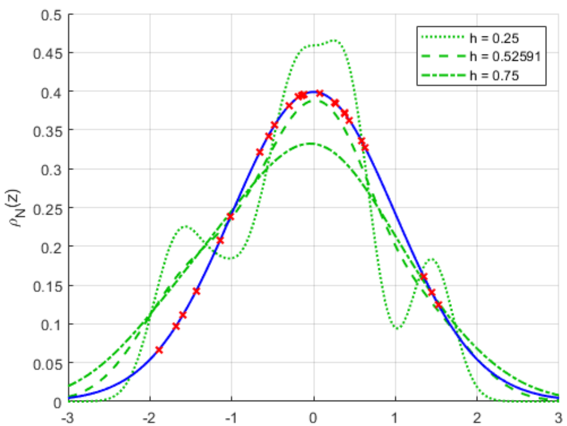
Mathematical Background

Definition: kernel density estimator

Let $\mathcal{Y} = \{y_1, \cdots, y_N\} \subseteq \mathbb{R}^n$ be independent and identically distributed samples of the random variable Y, which has an absolutely continuous distribution function with probability density function ϱ. Let K be a kernel function.
Then the kernel density estimator ϱ_N corresponding to the bandwidth $h \in (0, \infty)$ is defined as

$$\varrho_N(z) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{z - y_i}{h}\right).$$





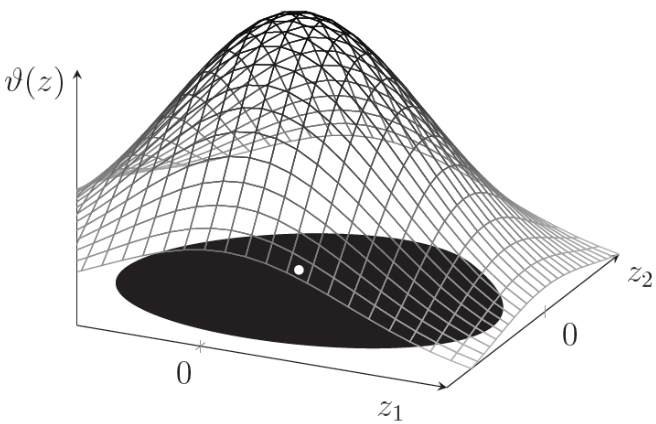
Mathematical Background

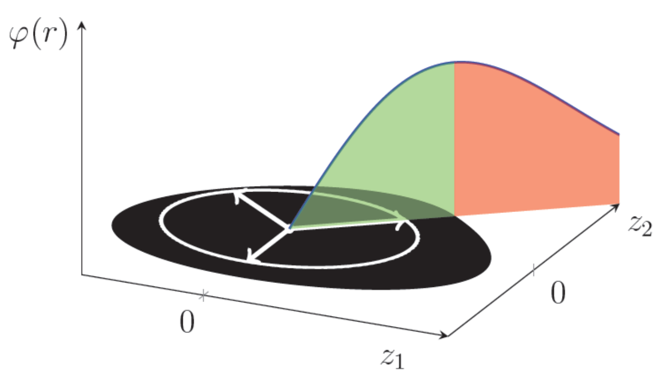
Theorem: Spheric radial decomposition (SRD)

Let $\xi \sim \mathcal{N}(0, R)$ be the n-dimensional standard Gaussian distribution with zero mean and positive definite correlation matrix R. Then, for any Borel measurable subset $M \subseteq \mathbb{R}^n$ it holds that

$$\mathbb{P}(\xi \in M) = \int_{\mathbb{S}^{n-1}} \mu_\chi\{r \geq 0 | rLv \in M\} d\mu_\eta(v),$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere in $\mathbb{R}^n$, μ_η is the uniform distribution on $\mathbb{S}^{n-1}$, μ_χ denotes the χ-distribution with n degrees of freedom and L is s.t. $R = LL^T$.





Outline

1) Probabilistic Constrained Optimization on Stationary Gas Networks

2) Probabilistic Constrained Optimization on Dynamic Gas Networks

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Stationary Gas Networks: Mathematical Modelling

- Consider a connected, directed, tree-structured graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- From the root the graph is numbered using breadth-first search or depth-first search

The stationary isothermal Euler equations for ideal gases:

quasilinear

$$q_x = 0,$$
$$\left(c^2\rho + \frac{q^2}{\rho}\right)_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}.$$

semilinear

$$q_x = 0,$$
$$c^2\rho_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}.$$

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The stationary isothermal Euler equations for ideal gases:

semilinear

$$q_x = 0,$$
$$c^2 \rho_x = -\frac{\lambda^F}{2D} \frac{q|q|}{\rho}.$$

solution

$$q(x) = k_1$$
$$p^2(x) = -\frac{\lambda^F}{c^2 D} (R_S T)^2 q(x)|q(x)|x + k_2,$$
$$k_1, k_2 \in \mathbb{R}.$$

Stationary Gas Networks: Mathematical Modelling

Boundary Conditions:

Inlet pressure

$$p^e(0) = p_0 \in \mathbb{R}_{\geq 0} \quad \forall e \in \mathcal{E}_+(v_0)$$

Gas outflow

$b_i \in \mathbb{R}_{\geq 0}$ represents the consumers gas demand at node v_i $(i = 1, \cdots, n)$

Coupling conditions at the nodes:

Conservation of mass

$$\sum_{e \in \mathcal{E}_-(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi = b^v + \sum_{e \in \mathcal{E}_+(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi \quad \forall v \in \mathcal{V} \backslash \mathcal{V}_0.$$

Continuity in pressure

$$p^{e_1}(L^{e_1}) = p^{e_2}(0) \quad \forall e_1 \in \mathcal{E}_-(v),\; e_2 \in \mathcal{E}_+(v).$$

Stationary Gas Networks: Mathematical Modelling

- Let $p \in \mathbb{R}^n$ be the vector of pressures at the nodes $v_1, \cdots, v_n$
- We assume box constraints for the pressures at the nodes: $p_i \in \left[p_i^{\min}, p_i^{\max}\right]$

Set of feasible loads

$M := \left\{ b \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{l} (p,q) \in \mathbb{R}^n \times \mathbb{R}^n \text{ satisfies:} \\ \bullet \text{ stationary semilinear isothermal Euler equations ,} \\ \bullet \text{ inlet pressure and gas outflow,} \\ \bullet \text{ conservation of mass and continuity in pressure,} \\ \bullet \text{ pressure bounds.} \end{array} \right\}$

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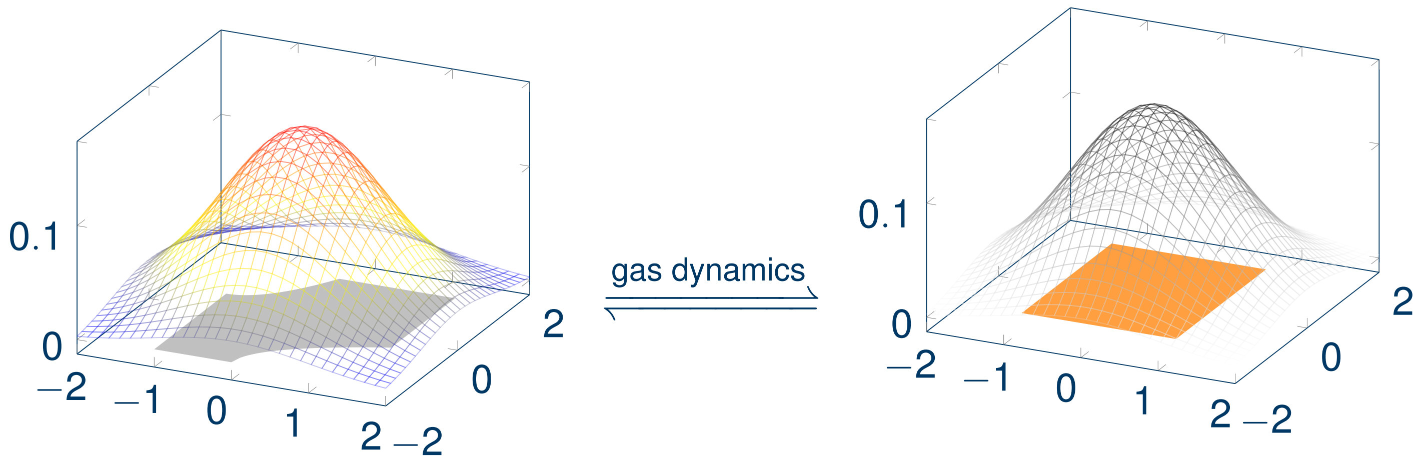
- Assume that the consumers gas demand is random in the sense, that there is a random variable
$$\xi_b \sim \mathcal{N}(\mu, \Sigma),$$
on an appropriate probability space. We identify b with the image $\xi_b(\omega)$ for $\omega \in \Omega$.

For a given inlet pressure, can we guarantee, that every consumer receives their demanded gas, s.t. the gas pressure in the network is neither too high nor too low, in at least α% of all scenarios?

Stationary Gas Networks: SRD vs KDE

Define $P_{\min}^{\max} := \otimes_{i=1}^{n} \left[p_i^{\min}, p_i^{\max}\right]$. Then we have $\mathbb{P}(b \in M) = \mathbb{P}\left(p \in P_{\min}^{\max}\right)$.



(a) Well-known distribution (colored), unknown set of feasible loads (gray)

(b) Unknown distribution (gray), well-known set of feasible pressures (orange)

Figure: Scheme of the SRD vs. scheme of the KDE

Stationary Gas Networks: Application of the SRD

Define a function $g : \mathbb{R}^n \to \mathbb{R}^n$, where $g_i(b)$ states the pressure loss from the root to node v_i.

Lemma 1

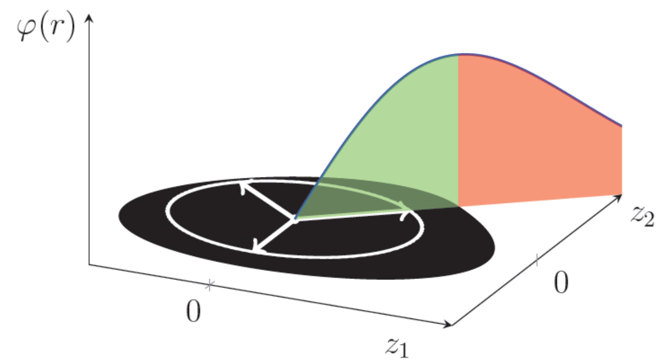
A load vector $b \in \mathbb{R}^n_{\geq 0}$ is feasible, i.e. $b \in M$, iff the following system of inequalities hold:

$$p_0^2 \leq \min_{k=1,\cdots,n} \left[(p_k^{\max})^2 + g_k(b) \right] ,$$

$$p_0^2 \geq \max_{k=1,\cdots,n} \left[(p_k^{\min})^2 + g_k(b) \right] .$$

Stationary Gas Networks: Application of the SRD

- Let $\mathcal{S} = \{ v_1, \cdots, v_{N_{\text{SRD}}} \}$ be a uniformly distributed sample of the unit sphere $\mathbb{S}^{n-1}$
- For $v \in \mathcal{S}$ identify the load vector with the one dimensional rays: $b_v(r) = rLv + \mu$
- Define the regular range: $R_{v,\text{reg}} := \{ r \geq 0 \mid b_v(r) \geq 0 \}$



- Use *Lemma 1* to compute the one dimensional sets $M_v = \bigcup_{j=1}^{\ell_v} [\underline{a}_{v,j}, \overline{a}_{v,j}]$
- Evaluate the χ-distribution: $\mu_\chi(M_v) = \sum_{j=1}^{\ell_v} F_\chi(\overline{a}_{v,j}) - F_\chi(\underline{a}_{v,j})$

probability via SRD

$$\mathbb{P}_{N_{\text{SRD}}}(b \in M) = \frac{1}{N_{\text{SRD}}} \sum_{v \in \mathcal{S}} \sum_{j=1}^{\ell_v} F_\chi(\overline{a}_{v,j}) - F_\chi(\underline{a}_{v,j})$$

Stationary Gas Networks: Application of the KDE

- Let $\mathcal{B} = \{ b^{S,1}, \cdots, b^{S,N_{\text{KDE}}} \} \subseteq \mathbb{R}^n_{\geq 0}$ be independent and identically distributed samples of the random variable ξ_b
- Let $\mathcal{P}_{\mathcal{B}} = \{ p(b^{S,1}), \cdots, p(b^{S,N_{\text{KDE}}}) \} \subseteq \mathbb{R}^n$ be the corresponding pressures at the nodes (also independent and identically distributed)

Gaussian kernel

$$K(x) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x_j^2 \right)$$

bandwidth matrix

$$H_{i,i} = h^2 \left(\Sigma_{N_{\text{KDE}}} \right)_{i,i}$$

$$h = \left(\frac{4}{(n+2) N_{\text{KDE}}} \right)^{\frac{1}{n+4}}$$

kernel density estimator ($h_j^2 := H_{j,j}$)

$$\varrho_{p,N_{\text{KDE}}}(z) = \frac{1}{N_{\text{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{z_j - p_j(b^{S,i})}{h_j} \right)^2 \right)$$

Stationary Gas Networks: Application of the KDE

$$\begin{aligned}
\mathbb{P}_{N_{\mathrm{KDE}}}(p \in P_{\min}^{\max}) &= \int_{P_{\min}^{\max}} \varrho_{p,N_{\mathrm{KDE}}}(z)\, dz \\
&= \int_{P_{\min}^{\max}} \frac{1}{N_{\mathrm{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathrm{KDE}}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z_j - p_j(b^{S,i})}{h_j}\right)^2\right) dz \\
&= \frac{1}{N_{\mathrm{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathrm{KDE}}} \prod_{j=1}^{n} \int_{p_j^{\min}}^{p_j^{\max}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z_j - p_j(b^{S,i})}{h_j}\right)^2\right) dz_j
\end{aligned}$$

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Gauss error function: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right)\, dt$

probability via KDE

$$\mathbb{P}_{N_{\text{KDE}}}(p \in P_{\min}^{\max}) = \frac{1}{N_{\text{KDE}} 2^n} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{n} \left[\operatorname{erf}\left(\frac{p^{\max} - p_j(b^{S,i})}{\sqrt{2}\, h_j}\right) - \operatorname{erf}\left(\frac{p^{\min} - p_j(b^{S,i})}{\sqrt{2}\, h_j}\right)\right]$$

SRD vs KDE: Advantages and Disadvantages

	+	−
SRD	• Powerful tool • Reduces dimension of integration	• Analytical solution required
KDE	• Almost always applicable	• Almost surely convergence • Curse of dimensionality

Stationary Gas Networks: Optimization

For $f : \mathbb{R}^n \to \mathbb{R}, p^{\max} \mapsto f(p^{\max})$ convex and $\alpha \in (0,1)$ consider

$$(O) \quad \begin{cases} \min_{p^{\max}\in\mathbb{R}^n} & f(p^{\max}) \\ \text{s.t.} & \mathbb{P}(b \in M(p^{\max})) \geq \alpha \\ & p^{\max} \geq p^{\min} \end{cases} \qquad (OA) \quad \begin{cases} \min_{p^{\max}\in\mathbb{R}^n} & f(p^{\max}) \\ \text{s.t.} & \mathbb{P}_{N_{\text{KDE}}}(b \in M(p^{\max})) \geq \alpha \\ & p^{\max} \geq p^{\min} \end{cases}$$

Lemma 2

- Assume that the set of admissible upper pressure bounds for (O) and (OA) is nonempty
- Assume that f is strictly monotonous increasing

Then there exists a solution $p^{*,\max}$ of (O) and $p^{*,\max}_{N_{\text{KDE}}}$ of (OA), s.t.

$$\mathbb{P}(b \in M(p^{*,\max})) = \alpha,$$
$$\mathbb{P}_{N_{\text{KDE}}}(b \in M(p^{*,\max}_{N_{\text{KDE}}})) = \alpha.$$

Stationary Gas Networks: Optimization

Define $g^\alpha(p^{\max})$ and $\mathcal{Z}$ ($g^\alpha_{N_{\text{KDE}}}$, $\mathcal{Z}_{N_{\text{KDE}}}$ analog) as

$$g^\alpha(p^{\max}) := \alpha - \mathbb{P}(b \in M(p^{\max})),$$
$$\mathcal{Z} := \{x \in \mathbb{R}^n \mid x \geq p^{\min} \text{ and } g^\alpha(x) = 0\}.$$

Theorem 3 part 1

Let a probability level $\alpha \in (0, 1)$, an inlet pressure p_0 and a lower pressure bound $p^{\min}$ be given.

- Assume that the set of admissible upper pressure bounds for (O) and (OA) is nonempty
- Assume that f is strictly monotonous increasing

Let $p^{*,\max}$ be a solution of (O).

Stationary Gas Networks: Optimization

Define $g^\alpha(p^{\max})$ and $\mathcal{Z}$ ($g^\alpha_{N_{\mathrm{KDE}}}$, $\mathcal{Z}_{N_{\mathrm{KDE}}}$ analog) as

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Theorem 3

Let $p^{*,\max}$ be a solution of (O).

- Assume that $p^{*,\max}$ is unique
- Assume that for all $x \in \mathcal{Z}$ there exist $d_1, d_2 \in \mathbb{R}^n \setminus \{0_n\}$ s.t. for all $\tau \in (0,1)$:

$$g^\alpha(x + \tau d_1) < 0 \quad \text{and} \quad g^\alpha(x + \tau d_2) > 0.$$

- Assume that there exist $\delta, \epsilon > 0$, s.t. for $p \in \mathcal{Z}$ with

$$\|p^{*,\max} - p\| > \delta/2 \quad \text{it holds} \quad |f(p^{*,\max}) - f(p)| > \epsilon$$

Then there exists N_{KDE} sufficiently large, s.t. the solution $p^{*,\max}_{N_{\mathrm{KDE}}}$ of (OA) is close to $p^{*,\max}$, i.e.

$$\|p^{*,\max} - p^{*,\max}_{N_{\mathrm{KDE}}}\| < \delta \qquad a.s.$$

Outline

1) Probabilistic Constrained Optimization on Stationary Gas Networks

2) Probabilistic Constrained Optimization on Dynamic Gas Networks

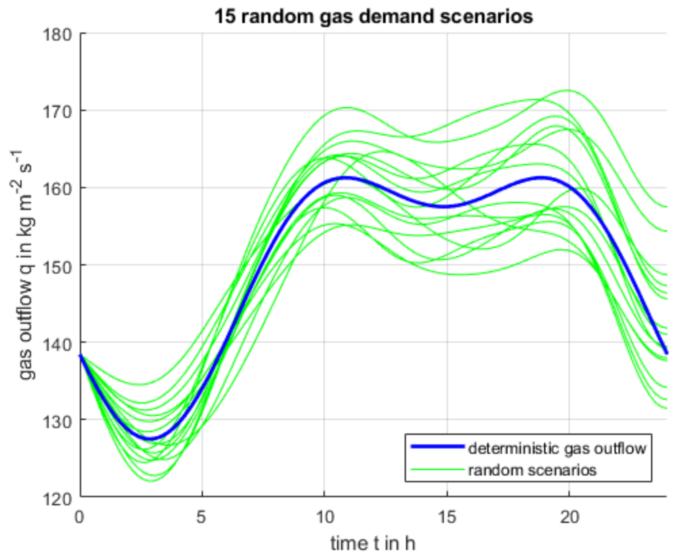
Time dependent Uncertainty

- Write a function f as Fourier series

$$f(t) = \sum_{m=0}^{\infty} a_m^0(f)\, \psi_m(t)$$

- For random variables $\xi_m \sim \mathcal{N}(1, \sigma)$ define

$$f^\omega(t) = \sum_{m=0}^{\infty} \xi_m(\omega)\, a_m^0(f)\, \psi_m(t)$$



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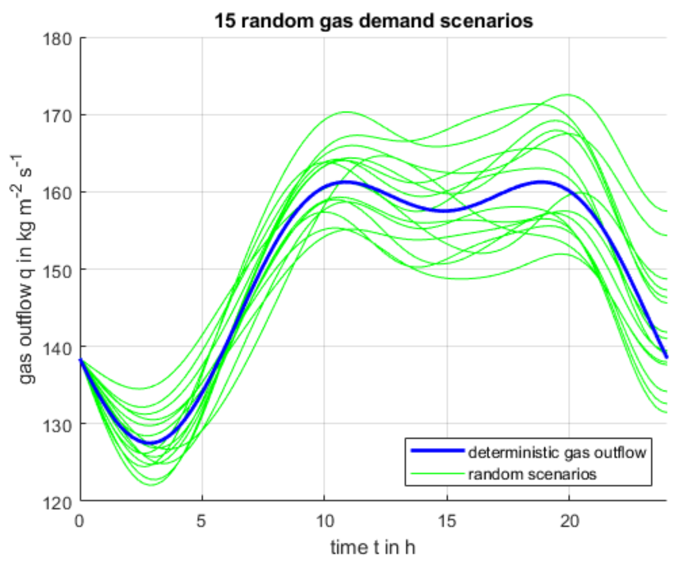


Time dependent probabilistic constraint

$$\mathbb{P}(\ b \in M(t)\ \forall t \in [0, T]\ ) \ \geq \ \alpha$$

„We want to guarantee that a percentage α of all possible random scenarios is feasible in every point in time $t \in [0, T]$."

Dynamic Gas Networks: Mathematical Modelling

- Consider a connected, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$.

isothermal Euler equations (for ideal gas)

$$(ISO) \quad \begin{cases} \rho_t + q_x = 0, \\ q_t + \left(c^2\rho + \frac{q}{\rho}\right)_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}. \end{cases}$$

set of feasible loads

$$M(t^*) := \left\{ \begin{array}{l} b : [0, T] \to \mathbb{R}^n \\ b_i \in \mathrm{Lip}([0, T]) \end{array} \middle| \begin{array}{l} \text{The solution of (ISO) with} \\ \quad \bullet \text{ initial conditions,} \\ \quad \bullet \text{ inlet density and gas outflow,} \\ \quad \bullet \text{ conservation of mass,} \\ \quad \bullet \text{ continuity in pressure,} \\ \text{satisfies box constraints for the density in } t^*. \end{array} \right\}$$

Dynamic Gas Networks: Application of the KDE

- Consider a sequence of random variables $(\xi_m)_{m\geq 0}$ with $\xi_m \sim \mathcal{N}(\mu, \Sigma)$
- For $m \geq 0$ let $\mathcal{A}_m = \{ a_m^{\omega,1}, \cdots, a_m^{\omega,N_{\text{KDE}}} \}$ be an independent and identically distributed sampling of the random variable ξ_m
- Let $\mathcal{B}_{\mathcal{A}} = \{ b^{\omega,1}(t), \cdots, b^{\omega,N_{\text{KDE}}}(t) \}$ be the corresponding sampling of the random boundary functions (also independent and identically distributed)
- Let $\mathcal{P}_{\mathcal{B}} = \{ \rho_{\text{out}}(t, b^{\omega,1}), \cdots, \rho_{\text{out}}(t, b^{\omega,N_{\text{KDE}}}) \}$ be the corresponding sampling of densities at the outflow nodes

$$\rho_{\text{out}}(t,\cdot) \in \mathcal{P}_{\text{min}}^{\text{max}} \;\forall t \in [0,T] \quad \Leftrightarrow \quad \min_{t\in[0,T]} \rho_{\text{out}}(t,\cdot),\ \max_{t\in[0,T]} \rho_{\text{out}}(t,\cdot) \in \mathcal{P}_{\text{min}}^{\text{max}}$$

- For $i = 1, \cdots, N_{\text{KDE}}$ let $\overline{\underline{\mathcal{P}}}_{\mathcal{B}} = \left[\left(\underline{\rho}_{\text{out}}(b^{\omega,i}),\ \overline{\rho}_{\text{out}}(b^{\omega,i}) \right)^{\top} \right]$ be the sampling of minimal and maximal densities.

probability via KDE

Apply KDE as in the stationary case to $2n$-dimensional sample

Dynamic Gas Networks: Application of the KDE

probability via kde

$$\mathbb{P}\left(\rho_{\text{out}}(t) \in \mathcal{P}_{\min}^{\max} \; \forall t \in [0, T] \right) =$$

$$= \frac{1}{N_{\text{KDE}}\, 2^{2n}} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{n} \left[\operatorname{erf}\left(\frac{\rho_j^{\max} - \underline{\rho}_{j,\text{out}}(b^{\omega,i})}{\sqrt{2}\, h_j^{\min}} \right) - \operatorname{erf}\left(\frac{\rho_j^{\min} - \underline{\rho}_{j,\text{out}}(b^{\omega,i})}{\sqrt{2}\, h_j^{\min}} \right) \right]$$

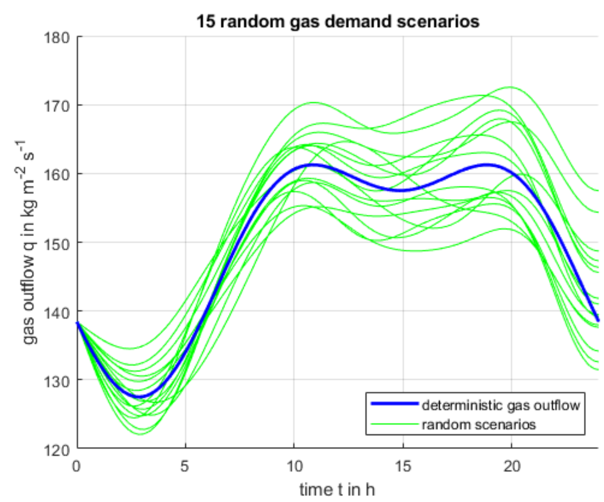
$$\cdot \left[\operatorname{erf}\left(\frac{\rho_j^{\max} - \overline{\rho}_{j,\text{out}}(b^{\omega,i})}{\sqrt{2}\, h_j^{\max}} \right) - \operatorname{erf}\left(\frac{\rho_j^{\min} - \overline{\rho}_{j,\text{out}}(b^{\omega,i})}{\sqrt{2}\, h_j^{\max}} \right) \right]$$

Optimization analog to the stationary case!

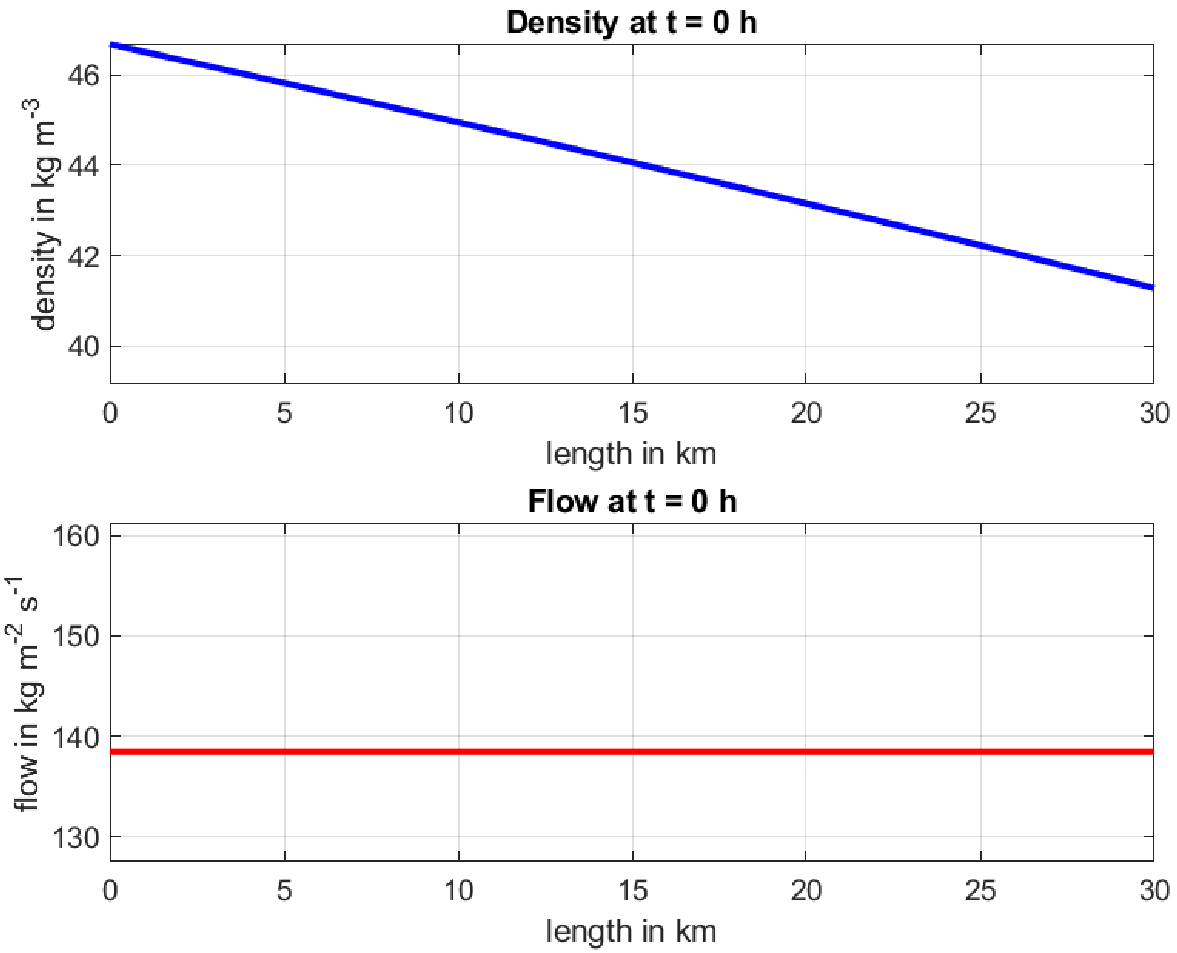
Dynamic Gas Networks: A Numerical Example

Consider (ISO) on a single edge:

Variable	Letter	Value	Unit
lower density bound	$\rho^{\min}$	34	kg/m^3
speed of sound in the gas	c	343	m/s
pipe friction coefficient	λ^F	0.1	
pipe diameter	D	0.5	m
pipe length	L	30	km
final time	T	24	h



Dynamic Gas Networks: A Numerical Example



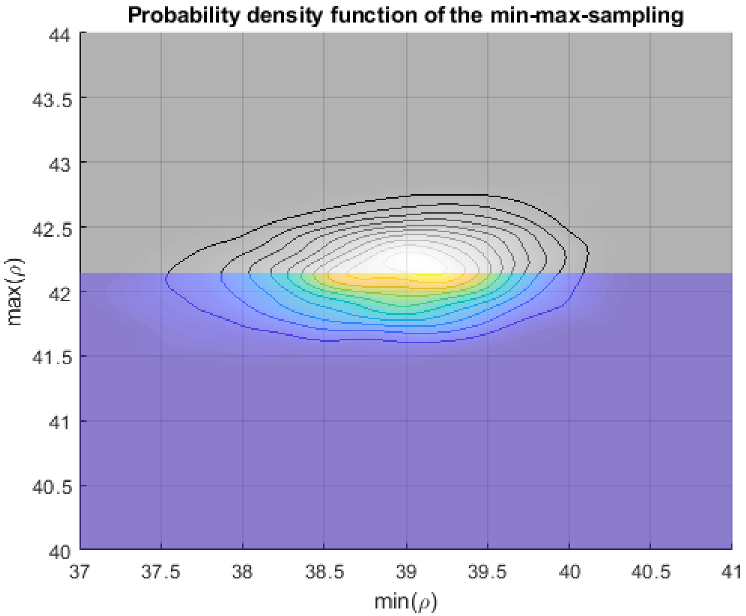
Dynamic Gas Networks: A Numerical Example

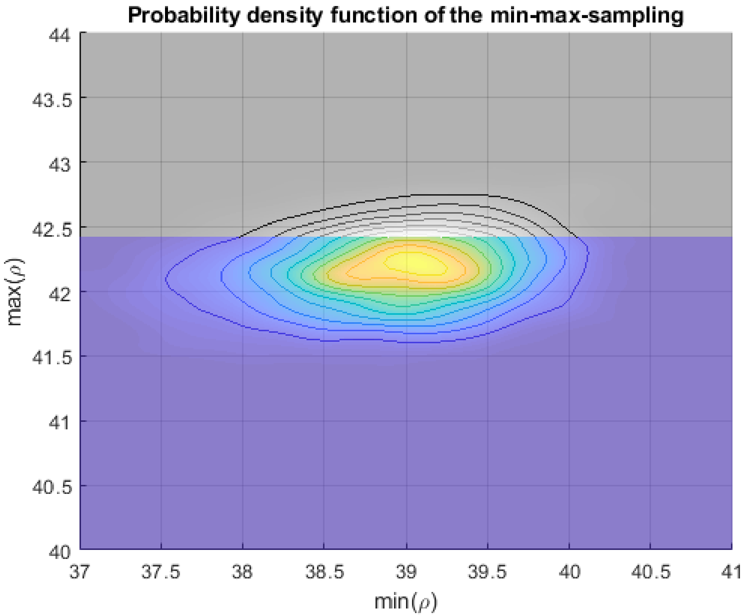
$$\min_{\rho_{\text{det}}^{\max}} \quad w\, f(p_{\text{det}}^{\max})$$
$$\text{s.t.} \quad \rho^{v_1}(t) \in \left[\rho^{\min}, \rho_{\text{det}}^{\max}\right] \quad \forall t \in [0,T]$$
$$\rho_{\text{det}}^{\max} \geq \rho^{\min}$$

$$\min_{\rho_{\text{prob}}^{\max}} \quad w\, f(p_{\text{prob}}^{\max})$$
$$\text{s.t.} \quad \mathbb{P}\left(\rho^{v_1}(t) \in \left[\rho^{\min}, \rho_{\text{prob}}^{\max}\right] \quad \forall t \in [0,T]\right) \geq 0.9$$
$$\rho_{\text{prob}}^{\max} \geq \rho^{\min}$$

$$\rho_{\text{det}}^{*,\max} = 42.15 \frac{\text{kg}}{\text{m}^3}$$

$$\rho_{\text{prob}}^{*,\max} = 42.49 \frac{\text{kg}}{\text{m}^3}$$





Summary

- We introduced the SRD and the KDE
- We computed the probability for a random load vector to be feasible using SRD and KDE respectively
- We discussed the advantages and disadvantages of both methods
- We showed the existence of optimal solutions for a certain probabilistic constrained optimization problem
- We showed that the optimal solution of the approximated problem is close to the optimal solution of the exact problem
- We extended the KDE approach to a dynamic setting

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THANK YOU