FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Some remarks on the turnpike property

Martin Gugat martin.gugat@fau.de
Chair Dynamics, Control and Numerics
Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
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CONTINUOUS OPTIMIZATION

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The classical turnpike phenomenon: An example with squared L2-tracking term

Examples with squared H1- and H2-tracking term

The finite-time turnpike phenomenon: An example with L1-tracking term

Optimal boundary control of a **hyperbolic 2x2 system**:
Motivating application: **Gas transport through pipelines**
(**TRR 154**)

Literature about the turnpike phenomenon

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The Turnpike Phenomenon

Consider a **dynamic optimal control problem** with finite time horizon and objective function of integral type: $\min \int_0^T \|y\|^2 + \|u\|^2$ s.t. $y(0) = y_0$, $y' = Ay + Bu$

Wikipedia: The **New Jersey Turnpike** Creative-Commons-Lizenz

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- If all the time-derivatives are set to zero and initial conditions and terminal conditions are canceled, this yields a **static optimal control problem**.

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- Turnpike results state **relations** between the ***static*** *optimal state/control* and the ***dynamic*** *optimal states/controls*.

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- Typically for large time intervals, close to its middle the ***dynamic*** *optimal states/controls* are **close** to the ***static*** *optimal state/control*.

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- If all the time-derivatives are set to zero and initial conditions and terminal conditions are canceled, this yields a **static optimal control problem**.
- Turnpike results state **relations** between the ***static*** *optimal state/control* and the ***dynamic*** *optimal states/controls*.
- Typically for large time intervals, close to its middle the ***dynamic*** *optimal states/controls* are **close** to the ***static*** *optimal state/control*.
- *In short*: The influence of the *initial data* and *terminal data* becomes **small** around $\frac{T}{2}$!

Wikipedia: The **New Jersey Turnpike** Creative-Commons-Lizenz

The Turnpike Phenomenon: A historical perspective

Very early references:

- JOHN VON NEUMANN (1937) *A Model of General Economic Equilibrium*
- FRANK RAMSEY (1928) *A Mathematical Theory of Saving*.

Later

- PAUL A. SAMUELSON (1976) *The periodic turnpike theorem*

And a quote from

- LW MCKENZIE (1986) *Optimal econ. growth, turnpike thms and comparative dynamics*:

"There is a fastest route between any two points;

and if the origin and destination are *close together* and far from the turnpike, the best route may not touch the turnpike.

But if origin and destination are *far enough apart*, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end"

Inhalt

An example with L^2 objective

A $1d$-example

Let $T > 0$, $\gamma > 0$ and $\lambda \neq 0$ be given. Consider the **dynamic** optimal control problem

$$\min \frac{1}{\pi} \int_0^T y^2(\tau) + \gamma\, u^2(\tau)\, d\tau$$

subject to

$$y(0) = y_0,\ \ y'(t) = \lambda\, (y(t) + u(t))\,,\ y(T) = y_0.$$

Without loss of generality we can put $T = 2\pi$.
Then we can write $y(t)$ as a FOURIERseries.

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With $y(t) = \frac{a_0}{2} + \sum_k a_k\, \cos(kt) + b_k\, \sin(kt)$,
$u(t) = \frac{v_0}{2} + \sum_k v_k\, \cos(kt) + w_k\, \sin(kt)$, we obtain

$\sum_k -k\, a_k\, \sin(kt) + k\, b_k\, \cos(kt) = \lambda\, \frac{a_0+v_0}{2} + \sum_k \lambda(a_k + v_k)\, \cos(kt) + \lambda(b_k + w_k)\, \sin(kt)$.
This yields $v_0 = -a_0$, $\lambda(a_k + v_k) = k\, b_k$ and $\lambda(b_k + w_k) = -k\, a_k$.

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For the objective value we obtain
$J = \frac{1+\gamma}{2}\, a_0^2 + \sum_k a_k^2 + b_k^2 + \gamma\, (\frac{k}{\lambda} b_k - a_k)^2 + \gamma\, (\frac{k}{\lambda} a_k + b_k)^2 = \frac{1+\gamma}{2}\, a_0^2 + \sum_k (a_k^2 + b_k^2)(1 + \gamma + \gamma\, \frac{k^2}{\lambda^2})$.

An example with $L2$ objective

A $1d$-example: The transformed optimal control problem is

$$\min \frac{1+\gamma}{2} a_0^2 + \sum_k (a_k^2 + b_k^2)(1 + \gamma + \gamma \frac{k^2}{\lambda^2})$$

subject to $\frac{a_0}{2} + \sum_k a_k = y_0$.

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The necessary optimality conditions yield a LAGRANGE multiplier μ such that

- $(1+\gamma)a_0 + \frac{1}{2}\mu = 0$ and for $k \in \{1, 2, 3, ...\}$

$$\begin{aligned} 2\,(1 + \gamma + \gamma \frac{k^2}{\lambda^2})\, a_k + \mu &= 0, \\ 2\,(1 + \gamma + \gamma \frac{k^2}{\lambda^2})\, b_k &= 0. \end{aligned}$$

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- Thus $b_k = 0$, $\frac{a_0}{2} = -\frac{1}{2(1+\gamma)} \frac{\mu}{2}$ and $a_k = -\frac{1}{1+\gamma+\gamma \frac{k^2}{\lambda^2}} \frac{\mu}{2}$.

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- This yields $y(t) = -\frac{\mu}{2} \left(\frac{1}{2(1+\gamma)} + \sum_k \frac{1}{1+\gamma+\frac{\gamma}{\lambda^2} k^2} \cos(k\,t) \right)$.

An example with $L2$ objective (continued)

The FOURIERseries of $\cosh(a(t-\pi))$ for $a \neq 0$ is

$$\cosh(a(t-\pi)) = \frac{\sinh(\pi\, a)}{\pi\, a} + \frac{2}{\pi}\sinh(\pi\, a)\sum_{k=1}^{\infty}\frac{a}{a^2+k^2}\cos(k\, t).$$

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Hence we have $\displaystyle \frac{\pi\, a}{2\ \sinh(\pi\, a)}\cosh(a(t-\pi)) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{a^2}{a^2+k^2}\cos(k\, t).$

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For the optimal state this yields with $a = |\lambda|\sqrt{1+\frac{1}{\gamma}}$:

$$y(t) = -\frac{\mu}{2}\left[\frac{1}{2(1+\gamma)} + \sum_{k}\frac{1}{1+\gamma+\frac{\gamma}{\lambda^2}k^2}\ \cos(k\, t)\right]$$

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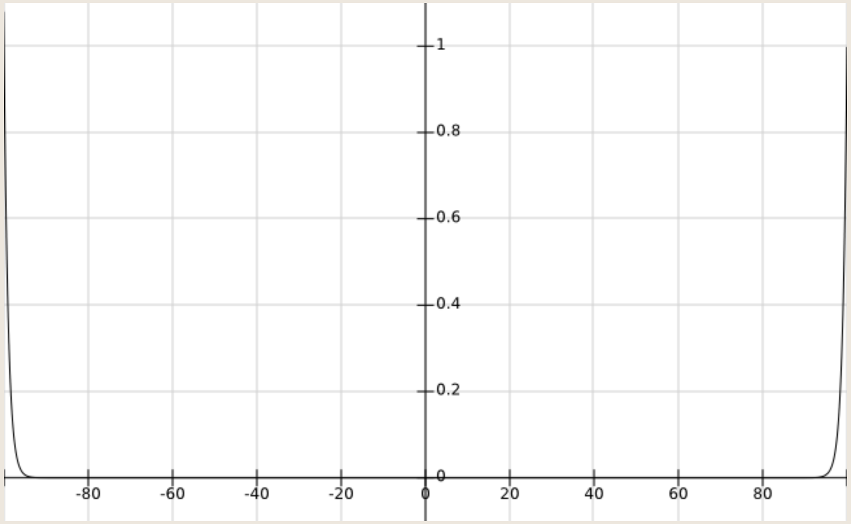
$$= -\frac{\mu}{2}\left[\frac{\pi}{2\ \sinh(\pi\,\lambda\sqrt{1+\frac{1}{\gamma}})}\ \frac{\lambda}{\sqrt{\gamma+\gamma^2}}\cosh\left(\lambda\sqrt{1+\frac{1}{\gamma}}\,(t-\pi)\right)\right]$$

An example with $L2$ objective (continued)

On a general time interval, for $t \in [0, T]$ we obtain an optimal state of the form

$$y(t) = \alpha \cosh\left(\lambda\sqrt{1+\frac{1}{\gamma}}\left(t-\frac{T}{2}\right)\right).$$



Let $y_0 = 1$, $T = 100$; $\frac{\cosh(t)}{\cosh(100)}$

Between -80 and 80, there is not much going on!

Inhalt

An example with $H1$ objective

A $1d$-example

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$J = \frac{1+\gamma}{2}\, a_0^2 + \sum_k (1 + \eta^2 k^2)\,(a_k^2 + b_k^2) + \gamma\,(\frac{k}{\lambda} b_k - a_k)^2 + \gamma\,(\frac{k}{\lambda} a_k + b_k)^2 =$
$\frac{1+\gamma}{2}\, a_0^2 + \sum_k (a_k^2 + b_k^2)\left(1 + \gamma + \left(\eta^2 + \frac{\gamma}{\lambda^2}\right)\, k^2\right).$

An example with $H1$ objective

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An H1 example (continued)

With the FOURIERseries (for $a \neq 0$)

$$\frac{\pi\, a}{2\ \sinh(\pi\, a)} \cosh(a(t-\pi)) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{a^2}{a^2+k^2} \cos(k\, t)$$

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for the optimal state we obtain with $a = |\lambda| \sqrt{\frac{(1+\gamma)}{\eta^2\lambda^2+\gamma}}$:

$$y(t) = -\frac{\mu}{2}\left[\frac{1}{2(1+\gamma)} + \sum_k \frac{1}{1+\gamma+\frac{\eta^2\lambda^2+\gamma}{\lambda^2}\, k^2} \cos(k\, t)\right] = -\frac{\mu}{2}\left[\frac{1}{2(1+\gamma)} + \frac{1}{1+\gamma}\sum_k \frac{\frac{\lambda^2(1+\gamma)}{\eta^2\lambda^2+\gamma}}{\frac{\lambda^2(1+\gamma)}{\eta^2\lambda^2+\gamma}+k^2} \cos(k\, t)\right]$$

$$= -\frac{\mu}{2}\left[\frac{1}{(1+\gamma)}\frac{\pi\, a}{2\;\sinh(\pi\, a)} \cosh\left(a\left(t-\pi\right)\right)\right]$$

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On a general time interval, for $t \in [0, T]$ we obtain an optimal state of the form

$$y(t) = \alpha \cosh\left(\lambda \sqrt{\frac{(1+\gamma)}{\gamma + \eta^2 \lambda^2}} \left(t - \frac{T}{2}\right)\right).$$

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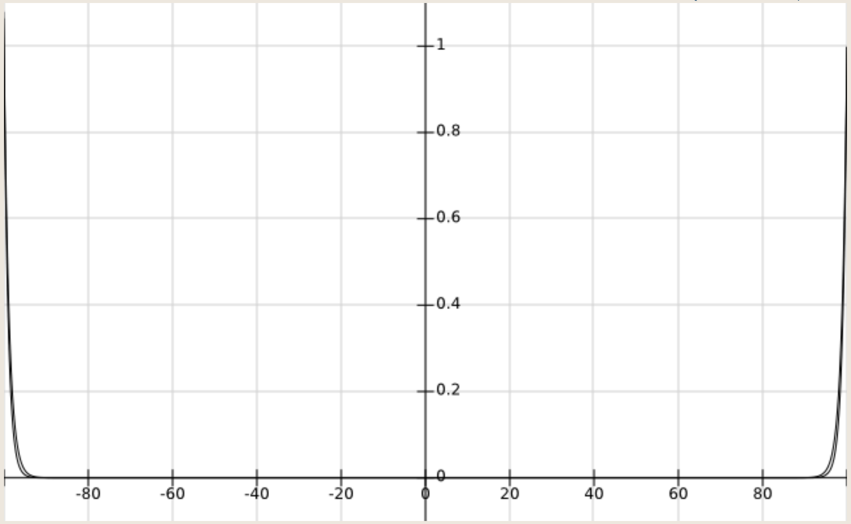
$\gamma = 1 = \eta$, $\lambda = \frac{1}{\sqrt{2}}$; $\frac{\cosh(t)}{\cosh(100)}$ and $\frac{\cosh(\frac{\sqrt{2}}{\sqrt{3}}\, t)}{\cosh(100\sqrt{2}/\sqrt{3})}$



For the L^2-case, we have $a_{L^2} = |\lambda|\sqrt{1 + \frac{1}{\gamma}} = |\lambda|\sqrt{\frac{(1+\gamma)}{\gamma}} > |\lambda|\sqrt{\frac{(1+\gamma)}{\gamma+\eta^2\lambda^2}} = a_{H^1}$.

An example with H2 objective

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Again for $T = 2\pi$ we write $y(t)$ as a FOURIERseries.

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$v_0 = -a_0$, $\lambda(a_k + v_k) = k\, b_k$ and $\lambda(b_k + w_k) = -k\, a_k$.
For the objective value we obtain
$J = \frac{1+\gamma}{2}\, a_0^2 + \sum_k (1 + \eta^2 k^4)\,(a_k^2 + b_k^2) + \gamma\,(\frac{k}{\lambda} b_k - a_k)^2 + \gamma\,(\frac{k}{\lambda} a_k + b_k)^2 =$
$\frac{1+\gamma}{2}\, a_0^2 + \sum_k (a_k^2 + b_k^2)\left(1 + \gamma + \eta^2\, k^4 + \frac{\gamma}{\lambda^2}\, k^2\right).$

An example with H2 objective

A $1d$-example: The transformed optimal control problem is

$$\min \frac{1+\gamma}{2} a_0^2 + \sum_k (a_k^2 + b_k^2)\left(\eta^2 k^4 + \frac{\gamma}{\lambda^2} k^2 + 1 + \gamma\right)$$

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The necessary optimality conditions yield a Lagrange multiplier μ such that

- $(1+\gamma)a_0 + \frac{1}{2}\mu = 0$ and for $k \in \{1, 2, 3, \ldots\}$

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- This yields $y(t) = -\frac{\mu}{2} \left[\frac{1}{2(1+\gamma)} + \sum_k \frac{1}{\eta^2 k^4 + \frac{\gamma}{\lambda^2} k^2 + 1 + \gamma} \cos(k\, t) \right]$. Thus y is twice continuously differentiable and $y'(0) = 0$. The *|curvature|* decays at 0!

An H2 example (continued)

What does the trajectory look like?

$$y(t) = \left[\frac{1}{2(1+\gamma)} + \sum_k \frac{1}{\eta^2 k^4 + \frac{\gamma}{\lambda^2} k^2 + 1 + \gamma} \cos(k\, t) \right].$$

If $\eta = \frac{\gamma}{2\sqrt{1+\gamma}\,\lambda^2}$ for $a = \frac{(1+\gamma)^{1/4}}{\sqrt{\eta}}$ this implies
$\eta\, y''(t) + \sqrt{1+\gamma}\, y(t) = \frac{1}{\sqrt{1+\gamma}} \frac{\pi\, a}{2\sinh(\pi\, a)} \cosh(a(t-\pi))$ [*Hidden turnpike* +trigonomet.?].

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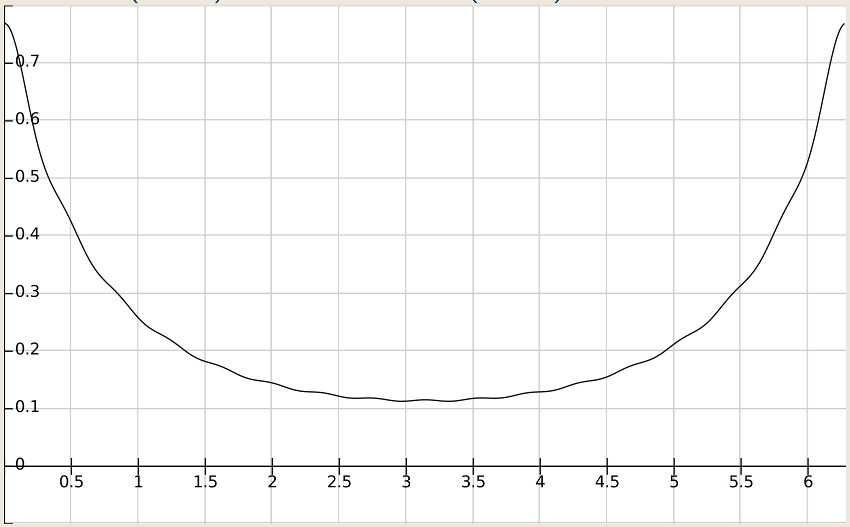
$\frac{1}{4} + \frac{1}{1+(1+1^2)^2}\cos(x) + \frac{1}{1+(1+2^2)^2}\cos(2x) + \cdots + \frac{1}{1+(1+16^2)^2}\cos(16x)$



Inhalt

An example with $L1$-tracking term

For $T \geq 1$ and $\gamma > 0$ we consider the problem

$$(\mathbf{OC})_T \begin{cases} \min\limits_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + \gamma\, |y(t)|\, dt \ \text{ subject to} \\ y(0) = -1, \\ y'(t) = y(t) + u(t). \end{cases}$$

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The solution of **(S)** is the **turnpike**! Here the **turnpike** is zero, $y^{(\sigma)} = 0$ and $u^{(\sigma)} = 0$.

An example with $L1$-tracking term

Solution of the problem without terminal constraint

- Variation of constants yields $y(t) = e^t \left[-1 + \int_0^t e^{-s}\, u(s)\, ds \right]$.

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- Consider the time-horizon T as a parameter.
  For $\tau \in (0, T]$, consider the **parametric optimal control problem** $(\mathbf{OC})_\tau$.
  As long as $y(\tau) \le 0$, we can *get rid of the absolute value* and thus of the non-smoothness!

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- So for sufficiently small $\tau > 0$, we consider

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- Integration by parts allows to transform the objective function:

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- For such a control $\int_0^s e^{-t}\, u(t)\, dt = 1$ holds if

$$\gamma = \frac{1}{\cosh(s) - 1}.$$

  Hence for $\tau = s$ we have $y(s) = 0$ and $u(s) = 0$. For $t > s$ we can continue with $u(t) = 0$ and obtain the optimal control for $(\mathbf{OC})_T$ for all $T \geq s$!

An example with $L1$-tracking term

Lemma (solution of $(\mathbf{OC})_T$, the problem without terminal conditions).

For $\gamma > 0$, define $s > 0$ as the value where $\cosh(s) = \frac{1}{\gamma} + 1$.
Assume that $T \geq s$. Define

$$\hat{u}(t) = \gamma(\mathrm{e}^{s-t} - 1)_+.$$

Then for the state $\hat{y}$ generated by $\hat{u}$ for $t \geq s$ we have $\hat{y}(t) = 0$.
Moreover, for all $t \in (0, T)$ we have $\hat{y}(t) \leq 0$.
The control $\hat{u}$ is the unique solution of $(\mathbf{OC})_T$.

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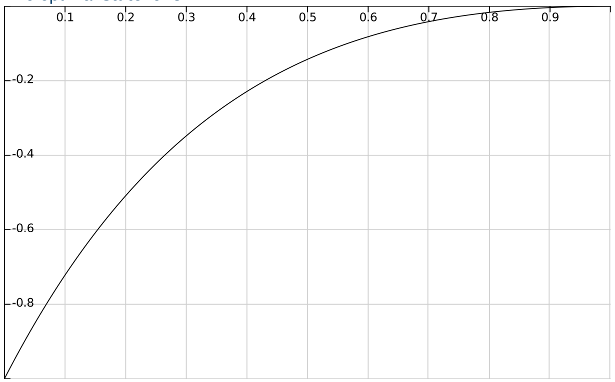
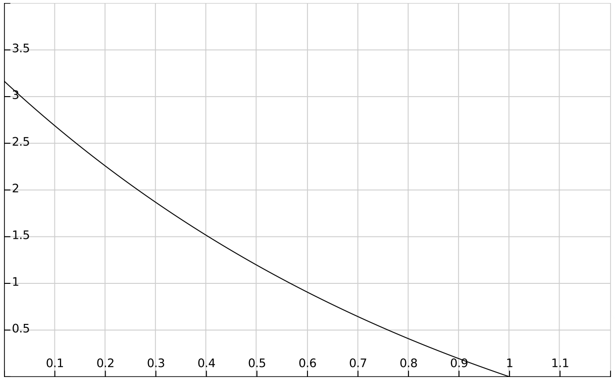
The optimal state is $y(t) = e^t \left[-1 + \gamma \left(\frac{e^s - e^{s-2t}}{2} + e^{-t} - 1 \right) \right]_- .$

The optimal **state** for $s = 1$

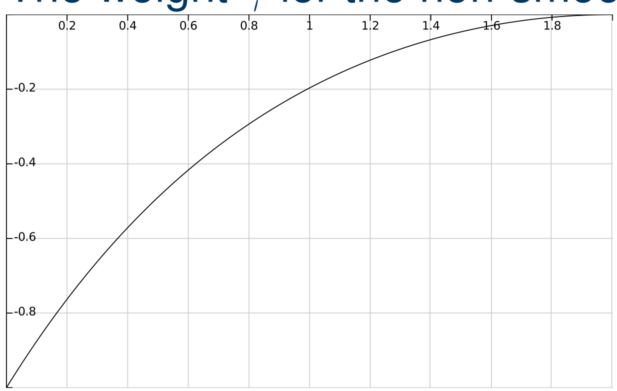


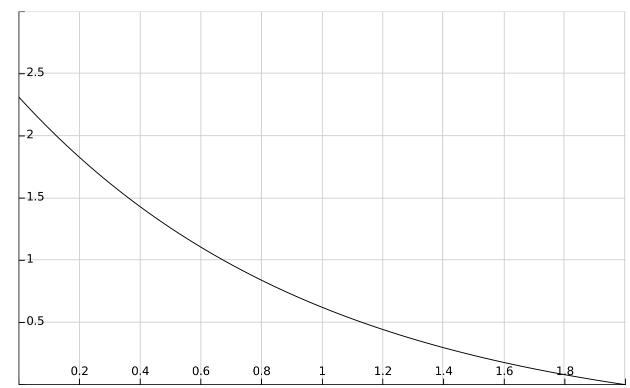
The optimal **control** for $s = 1$



An example with $L1$-tracking term

The optimal state and control for $s = 2$ and $\gamma = \frac{1}{\cosh(s)-1}$.
The weight γ for the non-smooth tracking term is made smaller than for $s = 1$.





An example with $L1$-tracking term

The exact turnpike phenomenon

- For sufficiently large T, due to the L^1-norm of y that appears in the objective function, the solution has a **finite–time turnpike structure**:
  The system is steered to zero in the *finite stopping time* s.

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- This time s is independent of T and *only depends on* γ.

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- For sufficiently large T, due to the L^1-norm of y that appears in the objective function, the solution has a **finite–time turnpike structure**:
  The system is steered to zero in the *finite stopping time* s.
- This time s is independent of T and *only depends on* γ.
- Both state and control remain at zero for all $t \in (s,\ T)$.

An example with $L1$-tracking term

For T sufficiently large, we can add a **terminal condition**:

For $T \geq 1$ and $\gamma > 0$ we consider the problem

$$(\textbf{FROMATOB})_T \begin{cases} \min\limits_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + \gamma\,|y(t)|\,dt \ \text{ subject to} \\ y(0) = -1,\ y(T) = 1, \quad y'(t) = y(t) + u(t). \end{cases}$$

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The problem decouples into $(\textbf{OC})_s$ (the problem **without terminal condition**) on $[0,\ s]$ and on the remaining time interval

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For the *starting time s* if $\cosh(T - s) - 1 = 1/\gamma$, the optimal control of $(\mathbf{END})_s$ is $u_2(t) = \gamma\,(1 - e^{s-t})_+$.

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Then we have $y(s) = 0$ and $y(t) = e^{t-T}[1 - \gamma(\frac{e^{s-T} - e^{T+s-2t}}{2} + e^{T-t} - 1)]$.

An example with $L1$-tracking term

The solution of $(\textbf{OC})_s$ reaches the turnpike after finite stopping time $s = s_{stop}$
The solution of $(\textbf{END})_s$ leaves the turnpike after the finite starting time $s = s_{start}$.
Hence if $s_{stop} \leq s_{start}$, the solutions can be **glued together** to solve $(\textbf{FROMATOB})_T$

if T is sufficiently large, e.g. $\frac{1}{\gamma} \in (0, \cosh(T/2) - 1)$
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From a given *initial state*, the optimal state is driven to the *turnpike* in finite time.

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Then it stays on the turnpike for a finite time interval.
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Text of the Song by *Kraftwerk*

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If the prescribed terminal state is different from the turnpike,
the state finally leaves the *turnpike* to reach the *target* state.

An example

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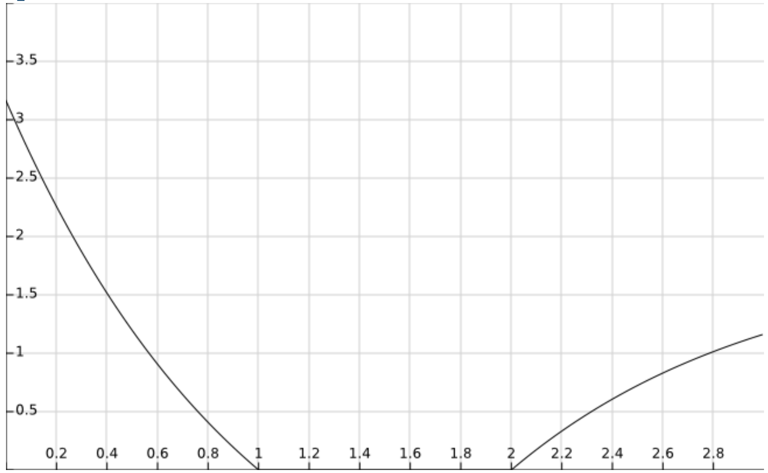
Assume that T is sufficiently large that $\cosh(T/2) - 1 > 1/\gamma$, that is $T > 2$.
Then for the optimal control and the optimal state we have

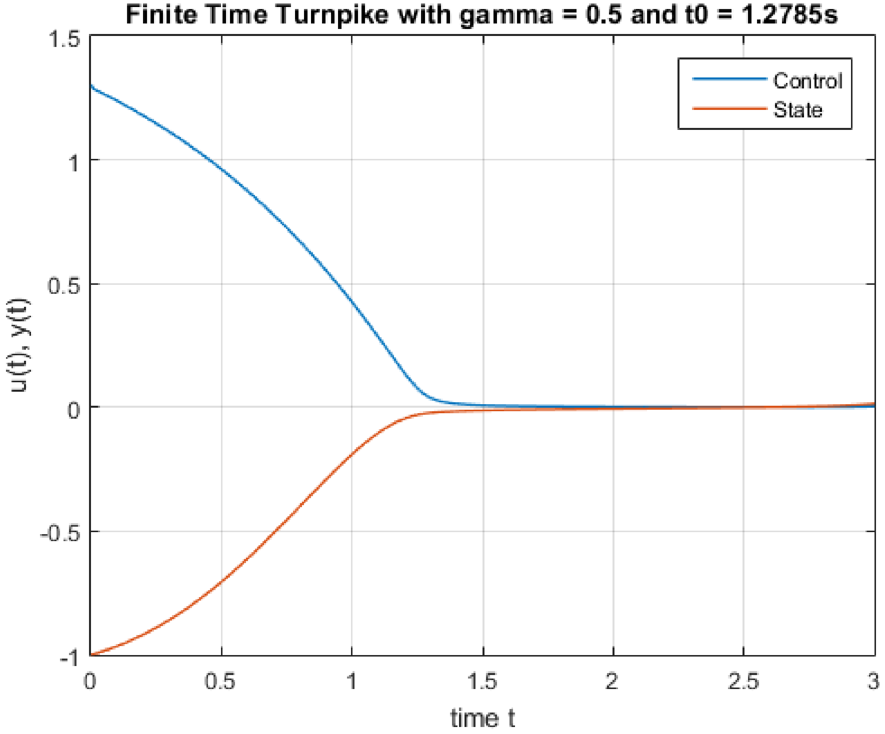
$$u(t) = 0,\ \ y(t) = 0 \ \text{ for all } t \in [1,\ \ T-1].$$

This is the **finite-time turnpike** situation.
On $[0,\ T]$ the control u is continuous and the state is continuously differentiable.

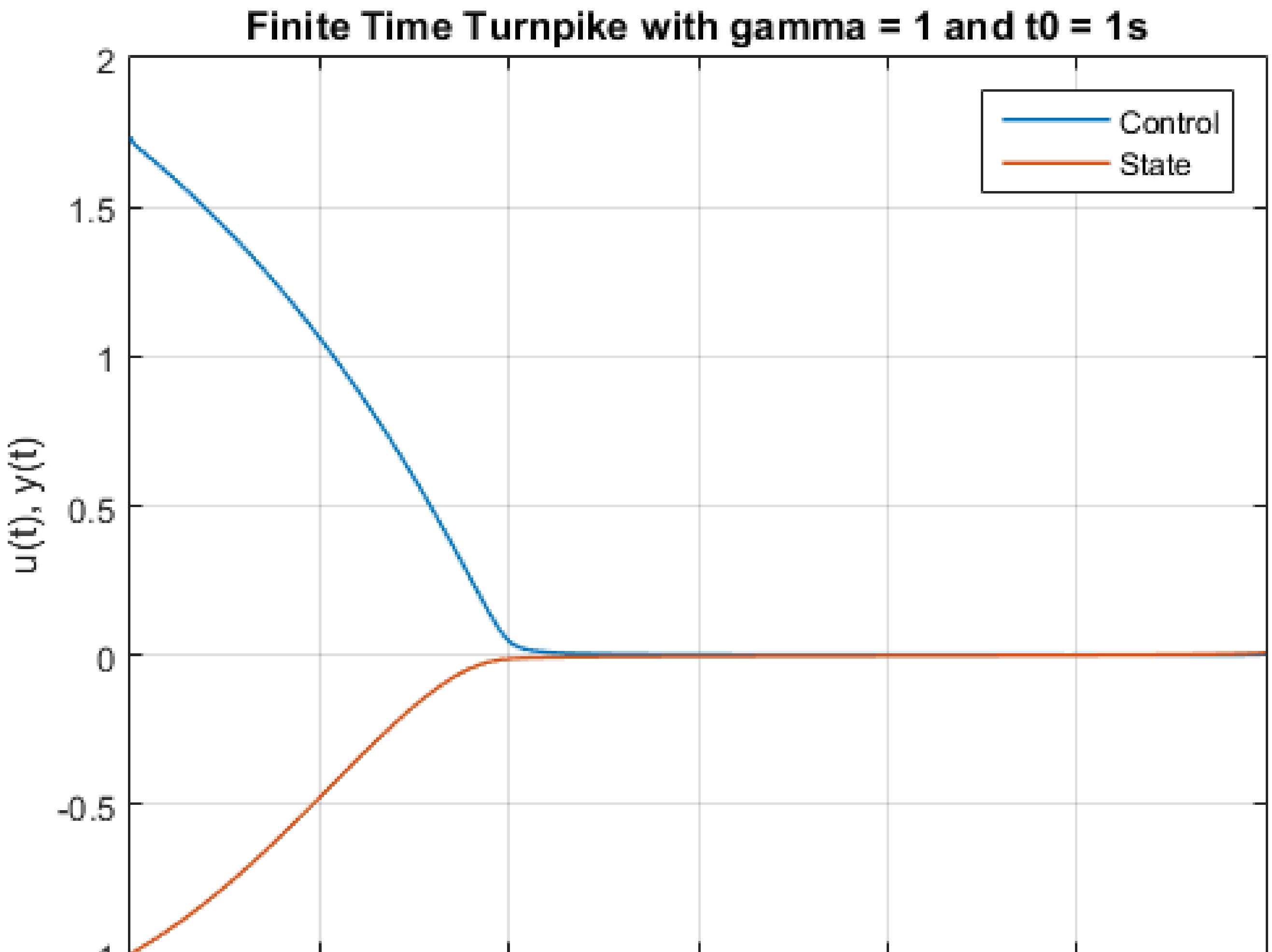


The optimal control

Numerical examples for $\min \int \frac{1}{2} u^2 + |u| + \gamma |y|$ with the nonautomous system $y'(t) = y(t) + e^t u(t)$, $y(0) = -1$ by Michael Schuster



Numerical examples for a nonautomous system by Michael Schuster: $y'(t) = y(t) + e^t u(t)$



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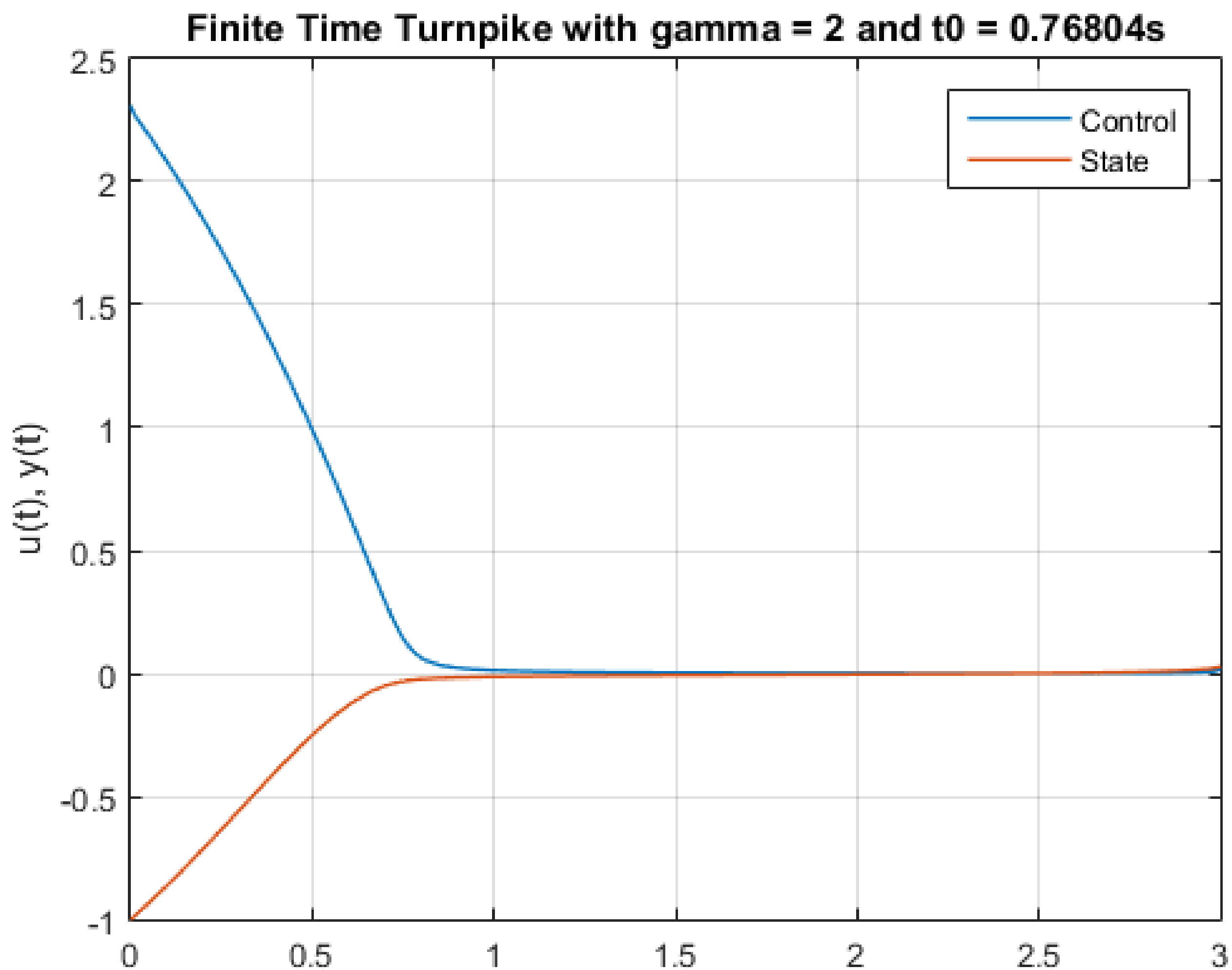

Paper on the Finite-Time Turnpike Phenomenon

*The **Finite-Time Turnpike Phenomenon** for Optimal Control Problems: Stabilization by Non-Smooth Tracking Terms,*

M. GUGAT, M. SCHUSTER, E. ZUAZUA, in
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- Contains also results for *infinite-dimensional* linear systems with L^∞ and L^2-norm tracking terms.
- Obviously, the finite-time turnpike phenomenon can only occur for systems that are *exactly controllable* in the sense that the turnpike is reachable.

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- Obviously, the finite-time turnpike phenomenon can only occur for systems that are *exactly controllable* in the sense that the turnpike is reachable.
- However, this also happens for systems that are *nodal profile exactly controllable* if the turnpike is prescribed through the *nodal profiles* in the objective function.

Inhalt

Transregio 154 Mathematical modelling, simulation and optimization using the example of gas networks

Project C03: Nodal control and the turnpike phenomenon (with RÜDIGER SCHULTZ)

Here the system **dynamics** on a **single pipe** is described by the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta}\frac{q|q|}{\rho} \end{cases}$$

or a similar (semilinear) model.
The gas pressure is increased at compressor stations.

See the results of DFG CRC 154:

TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

Inhalt

Literature about the turnpike phenomenon

- *Non-smooth tracking terms*: The **finite-time turnpike phenomenon** is possible.

Literature about the turnpike phenomenon

- *Non-smooth tracking terms*: The **finite-time turnpike phenomenon** is possible.

Smooth tracking terms: The classical **turnpike phenomenon**

1. **Exponential turnpike result** (with C_0 and $\mu > 0$ independent of T):

$$\|u(\tau) - u^\sigma\|^2 + \|y(\tau) - y^\sigma\|^2 \leq C_0 \left[\exp(-\mu\tau) + \exp(-\mu(T - \tau))\right]$$

For *pdes with distributed control* by PORRETTA and ZUAZUA (SICON 2013), and TRELAT, ZHANG, ZUAZUA (SICON 2018 in Hilbert space)
For *nonlinear ODEs* TRELAT, ZUAZUA in Journal of Differential Equations, 2015

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2. Another team: LARS GRÜNE and ANTON SCHIELA from Bayreuth, MANUEL SCHALLER and KARL WORTHMANN from Ilmenau and TIMM FAULWASSER from Dortmund.
   See for example *Abstract nonlinear sensitivity and turnpike analysis and an application to semilinear parabolic PDEs* ESAIM: COCV 27 (2021) 56.
   They often consider a characterization by *dissipativity inequalities* with a *storage function* and a *supply function*.

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3. ALEXANDER ZASLAVSKI (Technion). Many books!

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Weakest property: Measure turnpike property.

It holds if the ***measure*** *of the set where the distance between the optimal state and the turnpike is greater than a given bound* is uniformly bounded independently of the time horizon.

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Turnpike Property with Interior Decay, GUGAT, (MCSS 2021)

The **turnpike property with interior decay** requires that **there exist** $C_1 > 0$ **and** $\lambda_1 \in (0,1)$ such that for all $\lambda \in (0,1)$ and all T sufficiently large we have

$$\int_{\lambda T - \lambda_1 \lambda T}^{\lambda T + \lambda_1 (1-\lambda) T} \|\hat{y}_0(0, T, y_0, y_d)(s) - y^{(\sigma)}\|^2 \, ds \leq \frac{C_1}{\min\{\lambda, (1-\lambda)\}\, T}.$$

The subinterval of $[0, T]$ has the length $\lambda_1 T$.

Literature about the turnpike phenomenon

Strongest Property: **Exponential turnpike** as above and e.g. GUGAT, TRELAT, ZUAZUA, Optimal Neumann control for the 1D wave equation (2016).

Finite horizon $T \geq 2$, $\gamma > 0$. Define (P):

$$
\begin{cases}
\min\limits_{u \in L^2(0,T)} \int_0^T (y_x(t,0))^2 + \gamma\, u^2(t)\, dt \ \text{ subject to} \\
y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\
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The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially.

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For $t \in (0, 2)$, $k \in \mathbb{N}_0$, $t + 2k \leq T$:

$$\boxed{u(t+2k) = \left(z_\gamma^k - \frac{1}{z_\gamma^k} \right) \frac{1+z_\gamma}{1-z_\gamma} H(t).}$$

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This is an exponential turnpike structure!

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Inhalt

Conclusion

- If the optimal control problems have the turnpike property, *numerical methods* can be *started* by using the static optimal state/control as a starting point. The quality of this approximation is good close to $T/2$.

Thank you for your attention!

Weiningen, Schweiz