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THE STOCHASTIC FHN NEURON MODEL IN THE EXCITABLE REGIME POSSESSES A GLOBAL RANDOM PULLBACK ATTRACTOR

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Introduction

Noise is ubiquitous in neurons and its functional role is not yet fully understood. Stochastic analysis has thus emerged as an important tool to handle the overwhelming dynamical complexity of these cells and to gain a better understanding of their spiking and information processing mechanisms from the dynamics of their model equations. Here, we prove that under multiplicative Brownian motion, there exists a global random pullback attractor for the FitzHugh-Nagumo (FHN) neuron model (a paradigmatic model for biological neurons) in the excitable regime.

Structure of the stochastic FHN neuron model and its dynamics

$$\begin{cases} dv_t &= (v_t - \frac{v_t^3}{3} - w_t + I)dt = f(v_t, w_t)dt, \\ dw_t &= \varepsilon(v_t + \alpha - \beta w_t)dt + h(w_t)\circ dB_t = \varepsilon g(v_t,w_t)dt + h(w_t)\circ dB_t. \end{cases} \tag{1}$$

- $(v_t, w_t) \in \mathbb{R}^2$ represent the voltage v and the recovery current w of the neuron.
- $0 < \varepsilon \ll 1$ is the timescale separation parameter, I is the input stimulus, α is constant parameter, β a co-dimension one Hopf bifurcation parameter with the Hopf value at $\beta = \beta_h(\varepsilon)$.
- $\circ dB_t$ stands for the Stratonovich stochastic integral w.r.t the Brownian motion B_t with amplitude $h(w_t) = \sigma_0 w_t$ (multiplicative noise) and models the random opening and closing of the ion channels.
- In the adiabatic limit $\varepsilon \to 0$, the deterministic **critical manifold** $\mathcal{M}_0$ (red cubic in the figure) defining the set of equilibria of the layer problem associated to Eq.(1) is

$$\mathcal{M}_0 := \Big\{(v_t, w_t) \in \mathbb{R}^2 : f(v_t,w_t) = 0\Big\}. \tag{2}$$

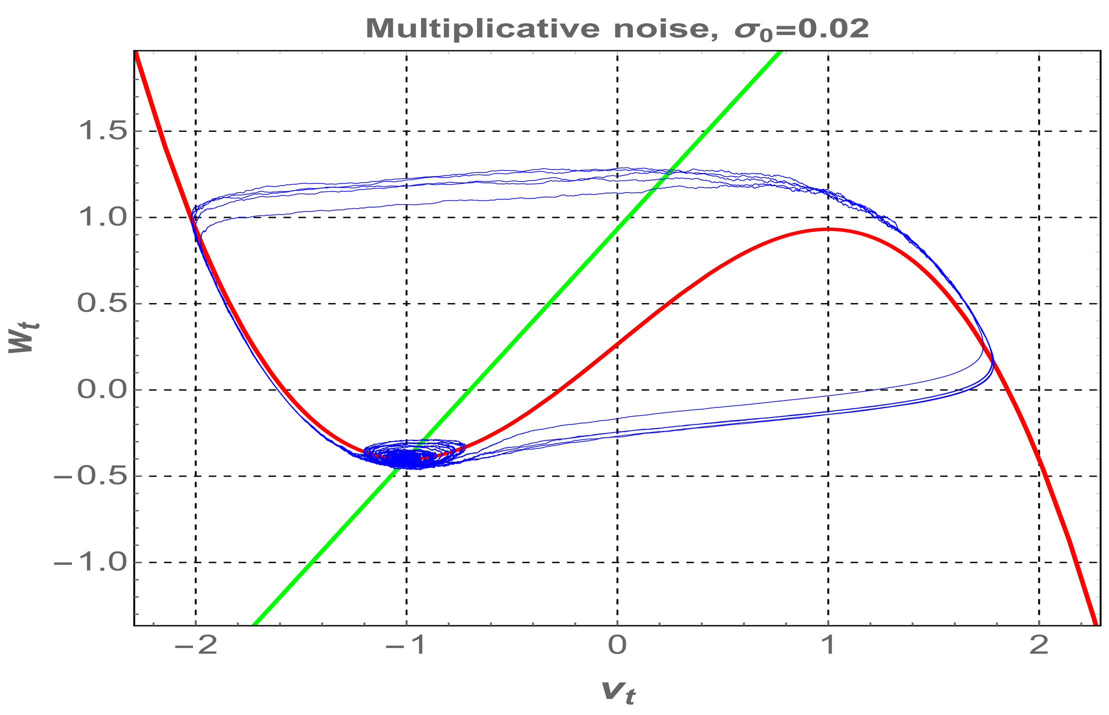
- $\mathcal{M}_0$ changes its stability at the **fold points** (v_f, w_f) defined by its Jacobian scalar

$$(v_f, w_f) := \Big\{(v_t, w_t) \in \mathbb{R}^2 : \partial_{v_t} f(v_t,w_t) = 0\Big\}. \tag{3}$$

- **The excitable regime** [2] of the neuron is characterized by existence of a **deterministic**, **unique**, and **stable equilibrium state** (v_e, w_e) given for Eq.(1) by

$$(v_e, w_e) := \Big\{(v_t, w_t) \in \mathbb{R}^2 : f(v_t,w_t) = g(v_t,w_t) = 0, (1/\beta - 1)^3 + 9(\alpha/\beta - I)^2/4 > 0, \\ \|\beta - \beta_h(\varepsilon)\| \le \delta > 0, v_e < v_f\Big\}. \tag{4}$$

- To investigate the asymptotic behavior of the neuron under the influence of noise, we first check the effect of the noise amplitude on the spiking. The figure shows a random trajectory (in blue) of Eq. (1) in the excitable regime around the **deterministic attractor** at $(v_e, w_e) = (-1.00125, -0.401665)$, i.e., the unique intersection point of $\mathcal{M}_0$ and the w_t-nullcline (green line in the figure) of Eq.(1).



The existence of a random pullback attractor

- With a Brownian motion B_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, (v_e, w_e) in Eq.(4) is no longer the equilibrium state of the neuron in Eq. (1). Instead, we need to find the global asymptotic state as a compact random set $A(\omega) \in \mathbb{R}^2$ depending measurably on $\omega \in \Omega$ such that A is invariant under φ, i.e., $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$, and attracts all other compact random sets $D(\omega)$ in the pullback sense, i.e.,

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)|A(\omega)) = 0, \tag{5}$$

where $d(B|A)$ is the Hausdorff semi-distance. Such a structure is called a **random pullback attractor** [1,3,4] with Ω chosen as $C^0(\mathbb{R},\mathbb{R})$ — the space of continuous real functions on $\mathbb{R}$ which are zero at zero and equipped with the compact open topology given by the uniform convergence on compact intervals in $\mathbb{R}$, $\mathcal{F}$ as $\mathcal{B}(C^0)$ — the associated Borel-σ-algebra, $\mathbb{P}$ as the Wiener measure, and $\theta_t(\cdot) : \Omega \to \Omega$ satisfying the group property, i.e., $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and is $\mathbb{P}$-preserving, i.e., $\mathbb{P}(\theta_t^{-1}(A)) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$, $t \in \mathbb{R}$.

Theorem [M.E. Yamakou et al. (2019)]

In the probabilistic setting above, there exists a unique solution of Eq. (1) which generates a random dynamical system. Moreover, Eq. (1) in the excitable regime, possesses a global random pullback attractor.

Proof

We introduce the transformation

$$\mathbf{Y}_t = (v_t, \bar{\omega}_t)^T := \begin{pmatrix} 1 & 0 \\ 0 & e^{-\sigma_0 z_t} \end{pmatrix} \mathbf{X}_t = T(z_t)\mathbf{X}_t, \tag{6}$$

where z_t is the unique stationary solution of the Ornstein-Uhlenbeck equation

$$dz_t = -z_t dt + dB_t. \tag{7}$$

Proof continued

This transforms system Eq. (1) into a random differential equation

$$\begin{cases} \dot{v}_t = v_t - \frac{v_t^3}{3} - e^{\sigma_0 z_t}\bar{\omega}_t + I, \\ \dot{\bar{\omega}}_t = e^{-\sigma_0 z_t}\varepsilon v_t + (\sigma_0 z_t - \varepsilon\beta)\bar{\omega}_t + \varepsilon\alpha e^{-\sigma_0 z_t}, \end{cases} \tag{8}$$

or equivalently

$$\dot{\mathbf{Y}}_t = G(z_t, \mathbf{Y}_t), \tag{9}$$

where G satisfies $G(z_t, 0) = (I, \varepsilon\alpha e^{-\sigma_0 z_t})^T$ and

$$\begin{aligned} \langle \mathbf{Y}_1 - \mathbf{Y}_2, G(z_t,\mathbf{Y}_1) - G(z_t,\mathbf{Y}_2)\rangle &= (v_1 - v_2)^2\Big[1 - \frac{1}{3}(v_1^2 + v_1v_2 + v_2^2)\Big] + (\sigma_0 z_t - \varepsilon\beta)(\bar{w}_1 - \bar{w}_2)^2 \\ &\quad + (\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})(v_1 - v_2)(\bar{w}_1 - \bar{w}_2) \\ &\le (v_1 - v_2)^2 - \frac{1}{12}(v_1 - v_2)^4 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2(v_1 - v_2)^2 \\ &\quad + (\sigma_0 z_t - \frac{\varepsilon\beta}{2})(\bar{w}_1 - \bar{w}_2)^2 \\ &\le -\frac{1}{12}(v_1 - v_2)^4 + (\sigma_0 z_t - \frac{\varepsilon\beta}{2})\|\mathbf{Y}_1 - \mathbf{Y}_2\|^2 \\ &\quad + \Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big](v_1 - v_2)^2 \\ &\le -\frac{1}{12}\Big((v_1 - v_2)^2 + 6\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]\Big)^2 \\ &\quad + 3\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]^2 \\ &\quad + (\sigma_0 z_t - \frac{\varepsilon\beta}{2})\|\mathbf{Y}_1 - \mathbf{Y}_2\|^2 \\ &\le 3\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]^2 \\ &\quad + (\sigma_0 z_t - \frac{\varepsilon\beta}{2})\|\mathbf{Y}_1 - \mathbf{Y}_2\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}\|\mathbf{Y}_t\|^2 &= 2\langle \mathbf{Y}_t - 0, G(z_t,\mathbf{Y}_t) - G(z_t,0)\rangle + 2\langle \mathbf{Y}_t, G(z_t,0)\rangle \\ &\le 3\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]^2 + (\sigma_0 z_t - \frac{\varepsilon\beta}{2})\|\mathbf{Y}_t\|^2 + 2\langle \mathbf{Y}_t, G(z_t,0)\rangle \\ &\le 3\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]^2 + \frac{4}{\varepsilon\beta}\|G(z_t,0)\|^2 + (\sigma_0 z_t - \frac{\varepsilon\beta}{4})\|\mathbf{Y}_t\|^2 \\ &\le 3\Big[1 + \frac{1}{2\varepsilon\beta}(\varepsilon e^{-\sigma_0 z_t} - e^{\sigma_0 z_t})^2 - \sigma_0 z_t + \frac{\varepsilon\beta}{2}\Big]^2 + \frac{4}{\varepsilon\beta}\Big[I^2 + \varepsilon^2\alpha^2 e^{-2\sigma_0 z_t}\Big] + (\sigma_0 z_t - \frac{\epsilon\beta}{4})\|\mathbf{Y}_t\|^2 \\ &\le p(z_t) + q(z_t)\|\mathbf{Y}_t\|^2. \end{aligned}$$

Hence by the comparison principle, $\|\mathbf{Y}_t\|^2 \le R_t$ whenever $\|\mathbf{Y}_0\|^2 \le R_0$ where R_t is the solution of

$$\dot{R}_t = p(z_t) + q(z_t)R_t, \tag{10}$$

which can be computed explicitly as

$$R_t(\omega, R_0) = e^{\int_0^t q(z_u(\omega))du}R_0 + \int_0^t p(z_s(\omega))e^{\int_s^t q(z_u(\omega))du}ds.$$

It is then easy to check that the vector field in Eq.(9) satisfies the local Lipschitz property and the solution is bounded and thus of linear growth on any fixed time interval $[0,T]$. Hence there exists a unique solution of Eq.(9) with initial condition, which also proves the existence and uniqueness of the solution of Eq.(8) and Eq. (1). Also, the solution generates a random dynamical system. On the other hand, observe that by the Birkhorff ergodic theorem, there exists almost surely

$$\lim_{t\to-\infty}\frac{1}{t}\int_t^0 q(z_u)du = \lim_{t\to-\infty}\frac{1}{t}\int_t^0 q(z(\theta_u\omega)) = E\Big[\sigma_0 z(\cdot) - \frac{\varepsilon\beta}{4}\Big] = -\frac{\varepsilon\beta}{4} < 0,$$

therefore there exists a unique stationary solution of Eq.(10) which can be written in the form

$$\bar{R}(\omega) = \int_{-\infty}^0 p(z_s(\omega))e^{\int_s^0 q(z_u(\omega))du}ds.$$

Moreover, $\|\mathbf{Y}_t(\omega,\mathbf{Y}_0)\|^2 \le \bar{R}(\theta_t\omega)$ whenever $\|\mathbf{Y}_0\|^2 \le \bar{R}(\omega)$ and

$$\limsup_{t\to\infty}\|\mathbf{Y}_t(\theta_{-t}\omega,\mathbf{Y}_0)\|^2 \le \limsup_{t\to\infty} R_t(\theta_{-t}\omega, R_0) = \bar{R}(\omega).$$

Hence, the ball $B(0, R(\omega))$ is actually forward invariant under the random dynamical system generated by Eq.(8) and is also a pullback absorbing set. By applying H. Crauel [3], there exists a random attractor for Eq.(8). Due to the fact that z_t is the stationary solution of Eq.(7), it is easy to see that the random linear transformation $T(z_t)$ given in Eq.(6) is *tempered* i.e.,

$$0 \le \lim_{t\to\infty}\frac{1}{t}\log\|T(z(t)\| = \lim_{t\to\infty}\frac{1}{2t}\log(1 + e^{-2\sigma_0 z_t}) \le \lim_{t\to\infty}\frac{1}{2t}(1 + 2\sigma_0|z_t|) = 0.$$

Therefore, it follows from [4] that systems Eq. (1) and Eq.(8) are conjugate under the tempered transformation in Eq.(6), hence there exists also a global random pullback attractor for Eq. (1) □.

Future research and perspectives

- Establish the corresponding theorem for an independent time-correlated α-stable Lévy motion with discontinuous probability distributions and infinite second moment.
- The structure and dynamics of random attractors are still open problems and their detailed understanding would certainly shed more light on the complex spiking and information processing mechanisms in biological neurons [1,2].

Relevant Publications

[1] M.E. Yamakou, T.D. Tran, L.H. Duc, J. Jost. *The stochastic FitzHugh-Nagumo neuron model in the excitable regime embeds a leaky integrate-and-fire model.* Journal of Mathematical Biology 79(2):509–532 (2019)

[2] M.E. Yamakou, J. Jost. *Weak-noise-induced transitions with inhibition and modulation of neural oscillations.* Biological Cybernetics 112, 445–463 (2018)

[3] H. Crauel, A. Debussche, F. Flandoli. *Random attractors.* Journal of Dynamics and Differential Equations 9(2):307–341 (1997)

[4] P. Imkeller, B. Schmalfuss. *The conjugacy of stochastic and random differential equations and the existence of global attractors.* Journal of Dynamics and Differential Equations 13(2):215–249 (2001)

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