FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

FACULTY OF SCIENCES

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CONTROL AND STABILIZATION OF GEOMETRICALLY EXACT BEAMS

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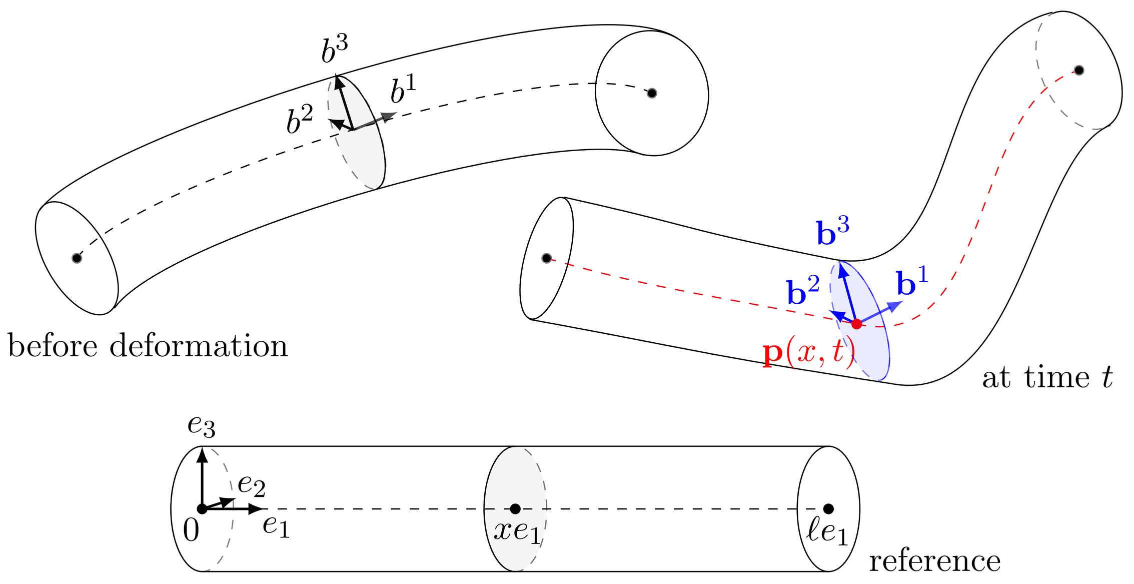
Introduction

There is a growing interest in modern highly flexible light structures – e.g. robotic arms, flexible aircraft wings, wind turbine blades, large spacecraft structures **[1]** – which exhibit *motions of large magnitude*, not negligible in comparison to the overall dimensions of the object. In engineering applications, there is also a clear need to control and eliminate vibrations in these structures.

To capture large motions, one needs so-called geometrically exact beam models (or networks of such beams), which are then nonlinear.

Dynamics of a geometrically exact beam



Two frameworks:

Quasilinear second-order (Reissner '81, Simo '85) *'Wave-like'*

Semilinear (quadratic) first-order hyperbolic (Hodges '03) *'Hamiltonian framework'*

Framework 1. The state is $(\mathbf{p}, \mathbf{R})$, expressed in some fixed coordinate system $\{e_j\}_{j=1}^3$,

centerline's position $\mathbf{p}(x,t) \in \mathbb{R}^3$

cross sections' orientation given by the columns $\mathbf{b}^j$ of $\mathbf{R}(x,t) \in \mathrm{SO}(3)$.

Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\partial_t \left[\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}\mathbf{M}v\right] = \partial_x \left[\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix} z\right] + \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ (\partial_x \widehat{\mathbf{p}})\mathbf{R} & \mathbf{0}\end{bmatrix} z, \qquad (1)$$

given $\mathbf{M}(x), \mathbf{C}(x) \in \mathbb{S}^6_{++}$ the mass and flexibility matrices and $\kappa(x) \in \mathbb{R}^3$ the curvature before deformation, and where v, s depend on $(\mathbf{p}, \mathbf{R})$:

$$v = \begin{bmatrix}\mathbf{R}^\intercal \partial_t \mathbf{p}\\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_t \mathbf{R})\end{bmatrix}, \quad z = \mathbf{C}^{-1}\begin{bmatrix}\mathbf{R}^\intercal \partial_x \mathbf{p} - e_1\\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_x \mathbf{R}) - \kappa\end{bmatrix}. \qquad (2)$$

Framework 2. The state is $y = \begin{bmatrix}v\\ z\end{bmatrix}$, expressed in the moving basis $\{\mathbf{b}^j\}_{j=1}^3$,

linear and angular velocities $v(x,t) \in \mathbb{R}^6$

internal forces and moments $z(x,t) \in \mathbb{R}^6$.

Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\end{bmatrix}\partial_t y - \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\partial_x y - \begin{bmatrix} 0 & 0 & \widehat{\kappa} & \mathbf{0}\\ 0 & 0 & \widehat{e}_1 & \widehat{\kappa}\\ \widehat{\kappa} & \widehat{e}_1 & 0 & 0\\ \mathbf{0} & \widehat{\kappa} & 0 & 0\end{bmatrix} y = - \begin{bmatrix}\widehat{v}_2 & \mathbf{0} & \mathbf{0} & \widehat{z}_1\\ \widehat{v}_1 & \widehat{v}_2 & \widehat{z}_1 & \widehat{z}_2\\ 0 & 0 & \widehat{v}_2 & \widehat{v}_1\\ 0 & 0 & \mathbf{0} & \widehat{v}_2\end{bmatrix}\begin{bmatrix}\mathbf{M}v\\ \mathbf{C}z\end{bmatrix}, \qquad (3)$$

denoting by v_1, z_1 and v_2, z_2 the first and last 3 components of v, z.

Notation. Cross-product: $\widehat{u}\,\zeta = u \times \zeta$ (thus $\widehat{u}$ skew-symmetric).
For any skew-symmetric $\mathbf{u} \in \mathbb{R}^{3\times 3}$, $\mathrm{vec}(\mathbf{u}) \in \mathbb{R}^3$ is such that $\mathbf{u} = \widehat{\mathrm{vec}(\mathbf{u})}$.
$\mathrm{SO}(3)$: rotation matrices. $\mathbb{S}^n_{++}$: positive definite symmetric matrices of size n.

Nonlinear transformation from (1) to (3)

One may move from one framework to the other via

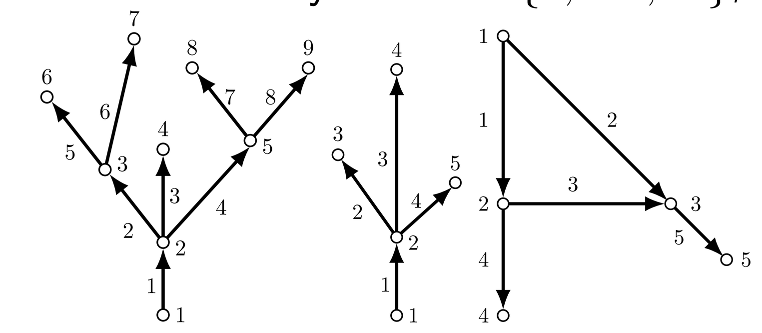
$$\mathcal{T}\colon \begin{cases} C^2_{x,t}\left(\mathbb{R}^3 \times \mathrm{SO}(3)\right) \longrightarrow C^1_{x,t}\left(\mathbb{R}^{12}\right)\\ (\mathbf{p}, \mathbf{R}) \longmapsto y \ \ \text{(defined by (2))}.\end{cases}$$

Theorem ([2], [4]). Under some compatibility conditions on the initial and boundary data of (1) and (3), $\mathcal{T}\colon E_1 \to E_2$ is bijective for some E_1, E_2 involving the last 6 equations in (3), the Dirichlet conditions (if any) and the (zero-order) initial conditions.

This allows us to translate some results on (3) into results on (1) (can be extended to networks).

Networks

N beams indexed by $i \in \mathcal{I} := \{1, \dots, N\}$, the state becomes $(\mathbf{p}_i, \mathbf{R})_{i\in\mathcal{I}}$ or $(y_i)_{i\in\mathcal{I}}$.



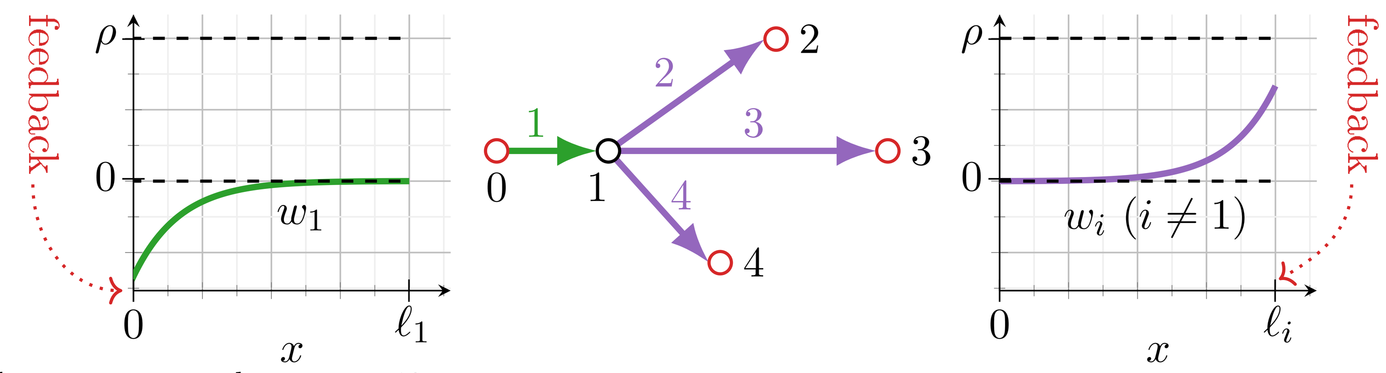
Interface conditions:

continuity of the centerlines, rigid joints

Kirchhoff condition (balance of forces and moments).

Local exponential stabilization for star-shaped networks

Boundary feedback control. At all simple nodes, we apply controls of the form $\nu_i z_i = -K_i v_i$ with $K_i \in \mathbb{S}^6_{++}$ (where $\nu_i(x)$ is the outward pointing normal).



Let $\mathbf{H}^k_x := \prod_{i\in\mathcal{I}} H^k(0, \ell_i; \mathbb{R}^{12})$.

Theorem ([2], [3]). For any $k \in \{1,2\}$, there exist $\beta, \eta > 0$ such that for all initial data $\mathbf{y}^0 = (y_i^0)_{i\in\mathcal{I}}$ small enough in $\mathbf{H}^k_x$, there exists a unique global in time solution $\mathbf{y} := (y_i)_{i\in\mathcal{I}} \in C^0([0,+\infty); \mathbf{H}^k_x)$ to the (3)-network, and

$$\|\mathbf{y}(\cdot,t)\|_{\mathbf{H}^k_x} \leq \eta e^{-\beta t}\|\mathbf{y}^0\|_{\mathbf{H}^k_x}, \quad \text{for all } t \in [0,+\infty).$$

Idea of the proof. (*Quadratic Lyapunov functional, Bastin, Coron '16*)
From the energy of the beam $\mathcal{E} = \sum_{i\in\mathcal{I}}\int_0^{\ell_i}\langle y_i\,, Q_i^{\mathcal{P}} y_i\rangle\, dx$, we build, for some $\rho > 0$, $w_i \in C^1([0,\ell_i])$ and $W_i = W_i(\mathbf{M}_i, \mathbf{C}_i)$:

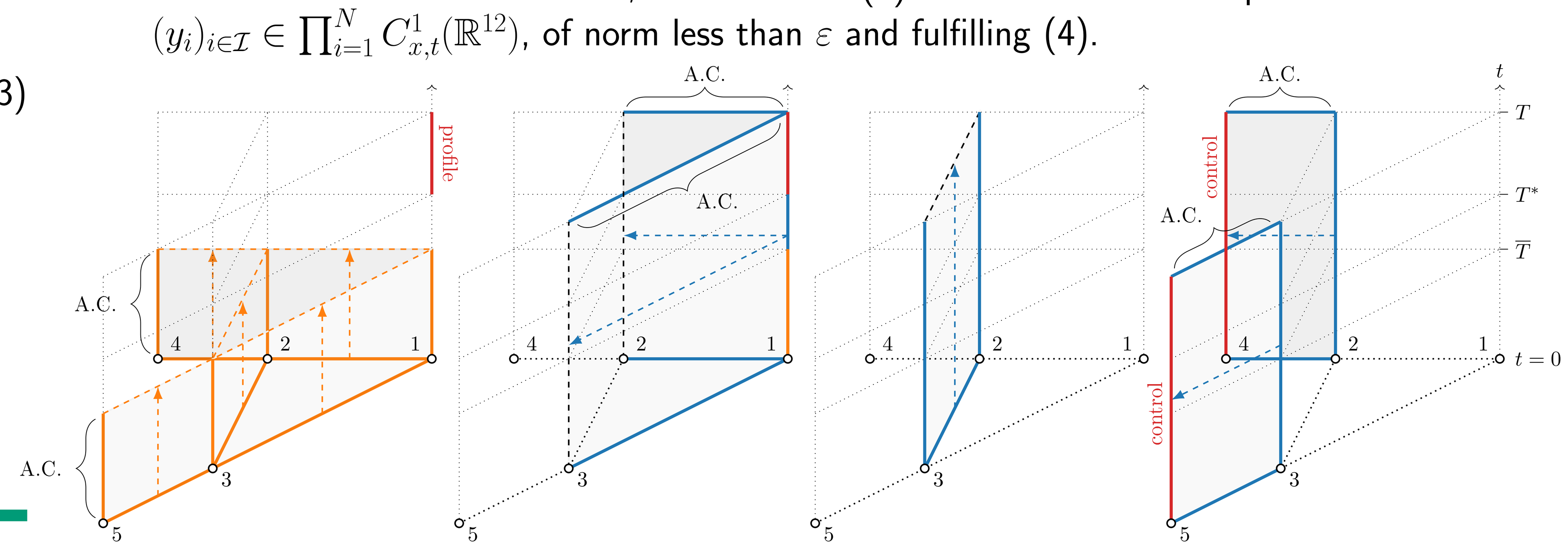
$$\mathcal{L} = \sum_{i\in\mathcal{I}}\sum_{\alpha=0}^{k}\int_0^{\ell_i}\left\langle \partial_t^\alpha y_i\,, \left(\rho\, Q_i^{\mathcal{P}} + w_i\begin{bmatrix}\mathbf{0} & W_i\\ W_i^\intercal & \mathbf{0}\end{bmatrix}\right)\partial_t^\alpha y_i\right\rangle dx.$$

Nodal profile control for networks with loops

For an A-shaped network, where z_i is controlled at the simple nodes $4, 5$, let $\overline{T}$ be the transmission time from the node 1 to the controlled nodes, and any $T > T^* > \overline{T}$.

Aim: *At the node* 1, *the state follows some given profiles in* $C^1([T^*, T]; \mathbb{R}^{12})$ *over the time interval* $[T^*, T]$. (4)

Theorem ([4]). There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for some $\delta > 0$, and for all initial and boundary data and nodal profiles with C^1 norm less than δ, there exist C^1 controls of norm less than ε, such that the (3)-network admits a unique solution $(y_i)_{i\in\mathcal{I}} \in \prod_{i=1}^N C^1_{x,t}(\mathbb{R}^{12})$, of norm less than ε and fulfilling (4).



Idea of the proof. (*Constructive method, Li, Rao '02,'03*)

1. forward problem until $\overline{T}$, and at node 1 connect this solution to the nodal profile to obtain 'initial data' for the next step,
2. series of forward and sidewise problems. Controls are given by the trace at nodes $4, 5$.

Outlook

- Global in time well-posedness (possibly adding structural damping), larger initial data.
- Stability: remove one of the controlled nodes, control less components of the state.
- Nodal profile control: result valid for any network, necessary and sufficient conditions.

Selected publications

[1] Géradin, M., Cardona, A. (2001). **Flexible Multibody Dynamics, A Finite Element Approach**, Wiley.

[2] Rodriguez, C., Leugering, G. (2020). **Boundary feedback stabilization for the intrinsic geometrically exact beam model.** SIAM J. Control Optim., 58(6), 3533–3558.

[3] Rodriguez, C. (2021). **Networks of geometrically exact beams: well-posedness and stabilization.** Math. Control Relat. Fields, doi:10.3934/mcrf.2021002.

[4] Leugering, G., Rodriguez, C., Wang, Y (2021). **Nodal profile control for networks of geometrically exact beams.** arXiv:2103.13064.

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Department of DATA SCIENCE

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