FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
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LONG TIME CONTROL

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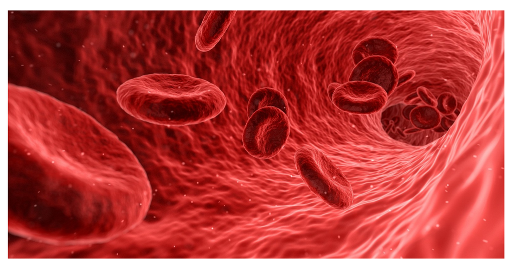
1. Why long time ?

Many applications of control impose considering problems in long time intervals *explicitly*: the treatment of *chronic* diseases, like hypertension and diabetes, for instance.



(a) Blood flow.

(b) Boat in a narrow channel (bilateral constraints).

FIGURE 1.

In others, large time is not required explicitly, but it is *implicitly needed* to fulfill the control requirements such as, for instance, state of control *constraints*, as in Figure 1.(b), where the boat across a narrow channel has to slow down to avoid crashing into the walls.

2. Minimal time to control under constraints

Consider the heat equation controlled from the boundary

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T)\times\Omega \\ y = u & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{1}$$

Our goal is to *control* this system to a steady state y_1

$$y(T,x) = y_1(x), \ x\in\Omega,$$

under the *constraint*

$$u(t,x)\geq 0, \ (t,x)\in(0,T)\times\partial\Omega.$$

The above problem is solvable under appropriate conditions on the data (y_0 and y_1 being positive steady states, for instance) and provided that time is sufficiently large. The method of proof combines local controllability, with a global "stair-case argument" (see [PZ2018]). Even though the constraints are of unilateral type, the large-time assumption is in fact necessary (e.g. [LTZ2017] and [PZ2018]). The *minimal controllability time* is positive

$$T_{\min} := \inf\left\{T>0 \;\middle|\; \exists\, u\geq 0, \ y(T,\cdot)=y_1\right\} > 0,$$

and, often, if $T_{\min}<\infty$.

We sketch the transposition method introduced in [PZ2018] (adaptable to the semilinear case). To fix ideas, suppose $y_0\equiv 0$. By transposition, the state y associated to a control u is characterized by the duality identity

$$\int_\Omega y(T,x)\varphi^0 dx + \int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n} d\sigma(x)dt = 0, \tag{2}$$

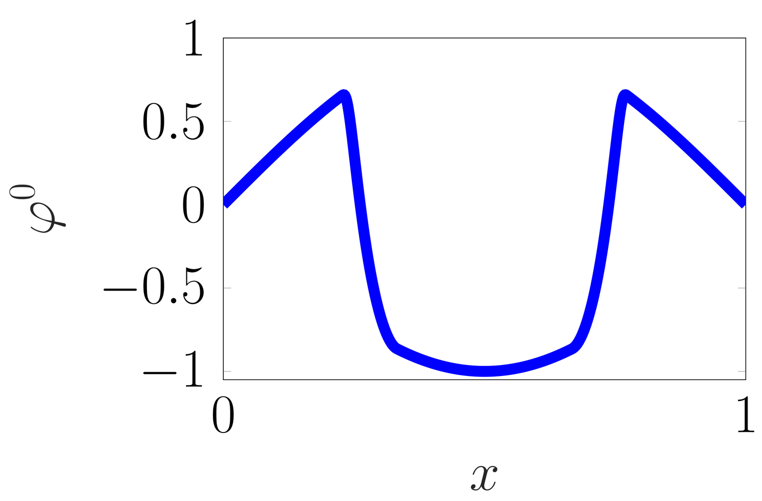
where φ is the solution to the *adjoint* problem:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x). & \text{in } \Omega \end{cases}$$

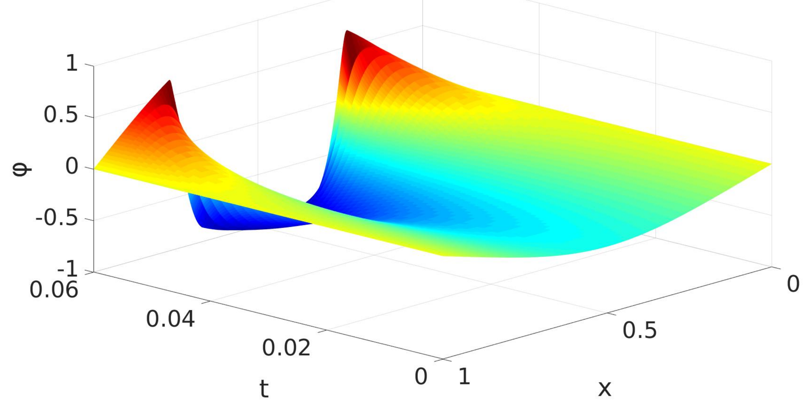
The existence of a special *final datum* φ^0 (Figure 2) and some time $T_0>0$, such that

$$\begin{cases} \frac{\partial\varphi}{\partial n}\leq 0 & \text{on } (0,T_0)\times\partial\Omega \\ \int_\Omega y_1\varphi^0 dx < 0, \end{cases}$$

leads to $T_{\min}\geq T_0>0$.



(a) Final datum.



(b) Backward evolution of the adjoint state.

FIGURE 2.

3. The turnpike phenomenon

Consider the time-evolution optimal control problem

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u) = \frac{1}{2}\int_0^T\int_\omega |u|^2dxdt + \frac{1}{2}\int_0^T\int_{\omega_0}|y-z|^2dxdt, \tag{$(OCP)_T$}$$

where:

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

The nonlinearity f is assumed to be C^3 *increasing*, with $f(0)=0$, whence with $u=0$ the solution decays exponentially, thus avoiding blow-up. $\omega\subseteq\Omega$ is the control domain, while $\omega_0\subset\Omega$ is the observation domain. By dropping the time-dependence, we get the steady problem

$$\min_{u_s\in L^2(\omega)} J_s(u_s) = \frac{1}{2}\int_\omega |u_s|^2dx + \frac{1}{2}\int_{\omega_0}|y_s-z|^2dx, \tag{$(OCP)_s$}$$

where:

$$\begin{cases} -\Delta y_s + f(y_s) = u_s\chi_\omega & \text{in } \Omega \\ y_s = 0 & \text{on } \partial\Omega. \end{cases}$$

By the Direct Method in the Calculus of Variations, there exists an optimal control $\overline{u}$ minimizing J_s. The corresponding optimal state is denoted by $\overline{y}$. If $\|z\|_{L^\infty(\omega_0)}$ is *small* enough, the steady problem admits a *unique* optimal control [PZ2016].

The control problem enjoys the *turnpike property* if for any optimal pair (u^T, y^T) for the time-evolution problem, there exists an optimal pair $(\overline{u},\overline{y})$ for the steady problem, s. t.

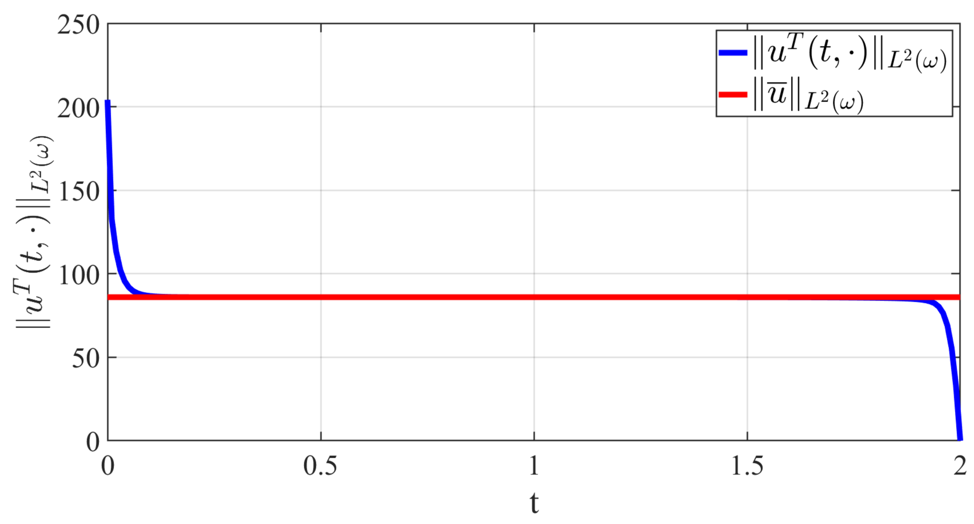


FIGURE 3. Norms of steady and time-evolution optimal controls.

$$\|u^T(t)-\overline{u}\|_{L^\infty(\omega)} + \|y^T(t)-\overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t}+e^{-\mu(T-t)}\right],$$

By *linearizing* the optimality system around the steady optimum and a *fixed point* argument, A. Porretta and E. Zuazua proved [PZ2016] the existence of $\delta>0$, such that the turnpike property holds, under the *smallness conditions*

$$\|y_0\|_{L^\infty(\Omega)}\leq\delta \text{ and } \|z\|_{L^\infty(\omega_0)}\leq\delta.$$

In [P2021], we have proved the existence of $\rho>0$ such that if

$$\|z\|_{L^\infty(\omega_0)}\leq\rho,$$

for every initial datum $y_0\in L^\infty(\Omega)$, the turnpike property holds.

4. Open problems and perspectives

We propose two open problems.

1. Develop complete turnpike theory for nonlinear control problems. For instance, for $(OCP)_T$, to the best of our knowledge, the case of *large* target z is still open. Note that, for a special target z, there exist (at least) two optimal controls for the steady problem $(OCP)_s$, when the domain Ω is a ball.

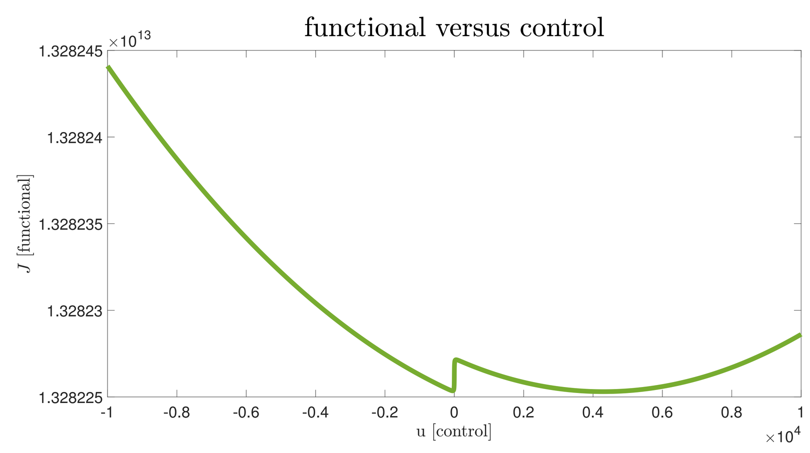


FIGURE 4. Plot of the steady functional with multiple minimizers versus constant controls.

2. Employ the turnpike phenomenon in value iteration algorithms of reinforcement learning. These algorithms can be initialized with the value of the functional at quasi-optimal turnpike strategies, consisting of three arcs: first, moving from the initial datum to the turnpike, then remaining there for long time and finally leaving at the end to match the terminal conditions.

Selected publications

[PZ2016] PORRETTA, A. & ZUAZUA, E. (2016). Remarks on long time versus steady state optimal control. *Mathematical Paradigms of Climate Science*, 67–89

[LTZ2017] LOHÉAC, J., TRÉLAT, E. & ZUAZUA, E. (2017) Minimal controllability time for the heat equation under unilateral state or control constraints. *Mathematical Models and Methods in Applied Sciences*, 27(9):1587–1644.

[PZ2018] PIGHIN, D. & ZUAZUA, E. (2018). Controllability under positivity constraints of semilinear heat equations. *Mathematical Control & Related Fields*, 8:935–964.

[P2021] PIGHIN, D. (2021). The turnpike property in semilinear control. *Accepted: ESAIM: COCV.*

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