Shape optimization and isoperimetric inequalities

Francesco Della Pietra

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”
Università degli Studi di Napoli Federico II

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Shape optimization

A shape optimization problem

To minimize (or maximize) a functional, depending on a shape Ω,

$$\mathcal{F}\colon \Omega \in \mathcal{U}_{ad} \mapsto \mathbb{R}$$

among all the sets Ω living in a class $\mathcal{U}_{ad}$ of *admissible shapes*.

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In many applications, the functional $\mathcal{F}(\Omega)$ depends on Ω via a *state function* u_Ω, which arises as the solution of a partial differential equation given in Ω.

Isoperimetric inequalities and symmetrization

Isoperimetric inequalities

The term *isoperimetric inequalities* is used in literature to identify inequalities which arise from minimum or maximum problems in which not necessarily perimeter and volume are involved. For example, they can regard geometric quantities as the diameter, general notion of perimeter, or physical quantities as capacity, eigenvalues of boundary value problems, torsional rigidity

Isoperimetric inequalities and symmetrization

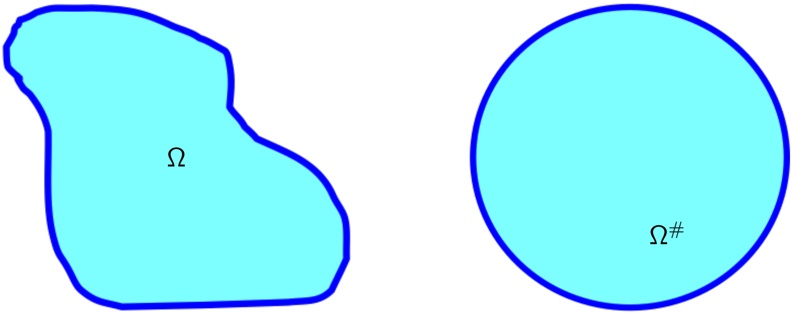
Isoperimetric inequalities

The term *isoperimetric inequalities* is used in literature to identify inequalities which arise from minimum or maximum problems in which not necessarily perimeter and volume are involved. For example, they can regard geometric quantities as the diameter, general notion of perimeter, or physical quantities as capacity, eigenvalues of boundary value problems, torsional rigidity

Symmetrization

A symmetrization procedure consists, roughly speaking, in transforming a mathematical object (*e.g.* a set, a function) in another one "more symmetric" so that it can preserve some property of the original object

Symmetrization of sets



$$|\Omega| = \left|\Omega^{\#}\right|$$

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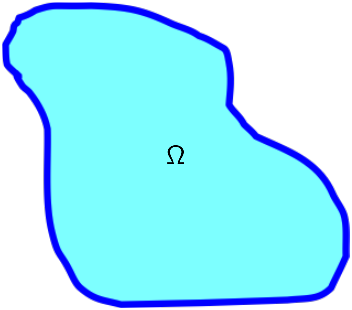

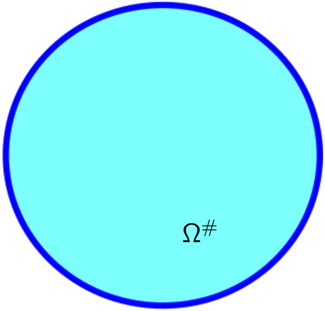

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Among all the open sets $\Omega \subseteq \mathbb{R}^N$ with fixed volume, the ball has the smallest perimeter.

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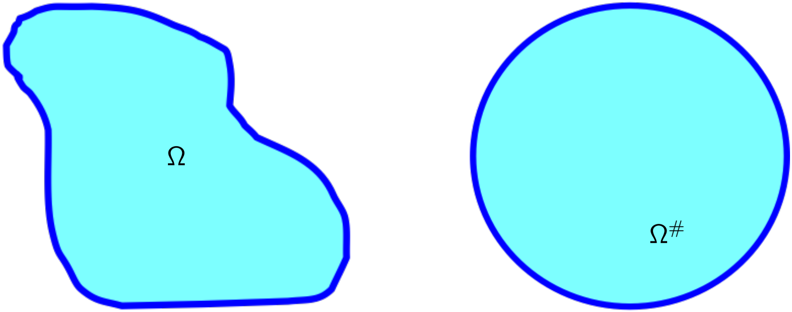

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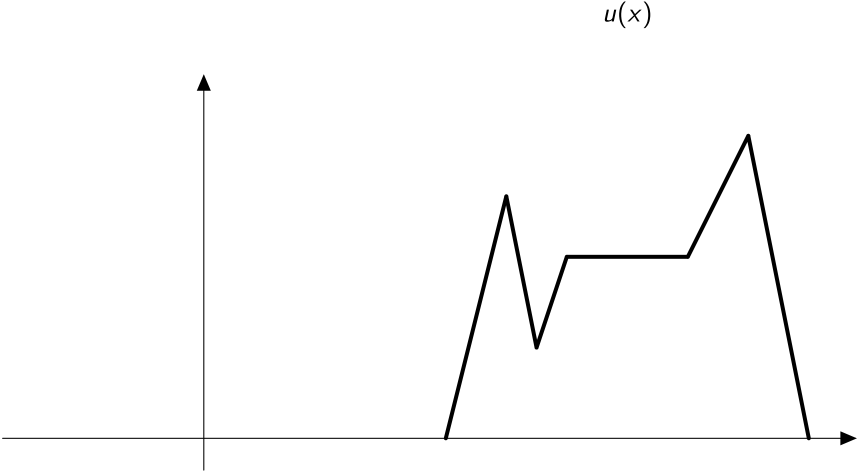
Among all the open sets $\Omega \subseteq \mathbb{R}^N$ with fixed volume, the ball has the smallest perimeter.

$$\min_{|\Omega|=k} Per(\Omega) = Per(\Omega^{\#})$$

Symmetrization of functions



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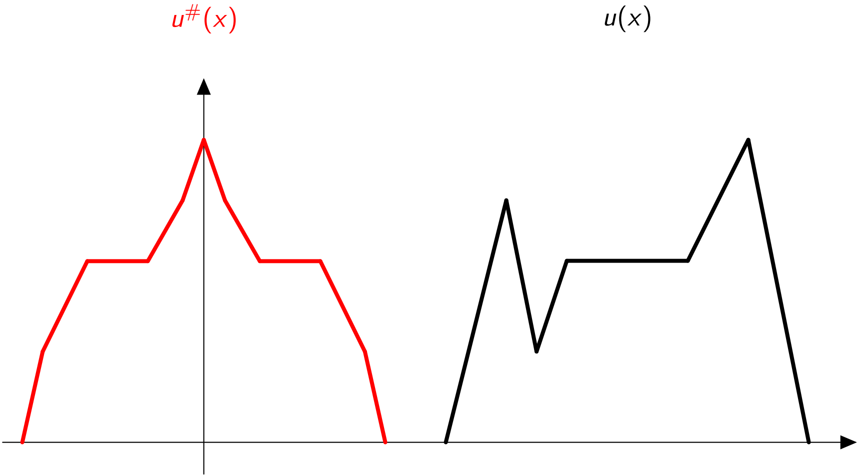

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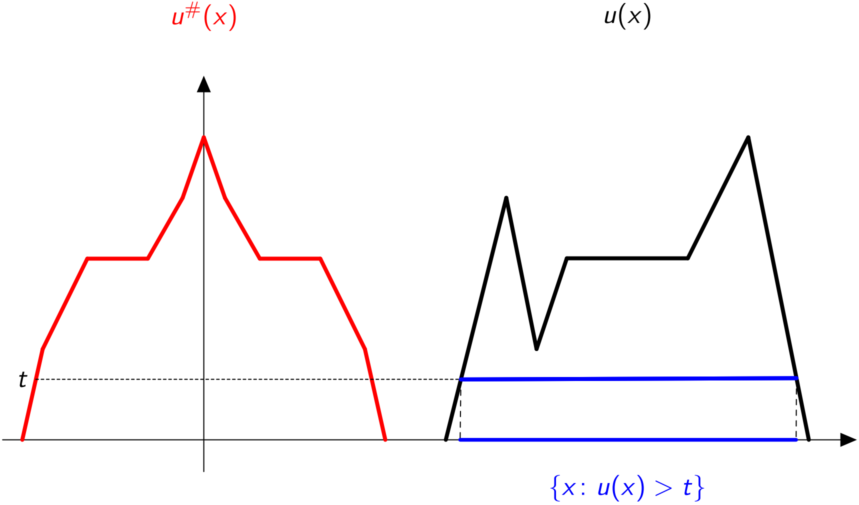

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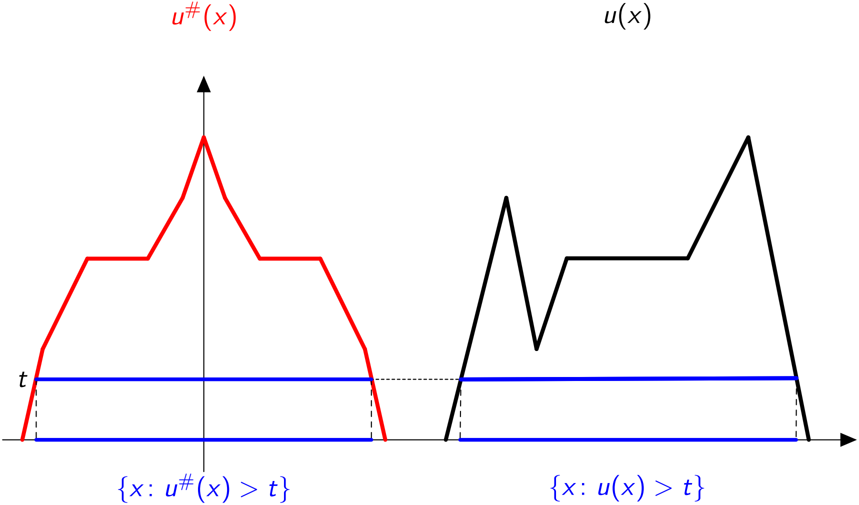

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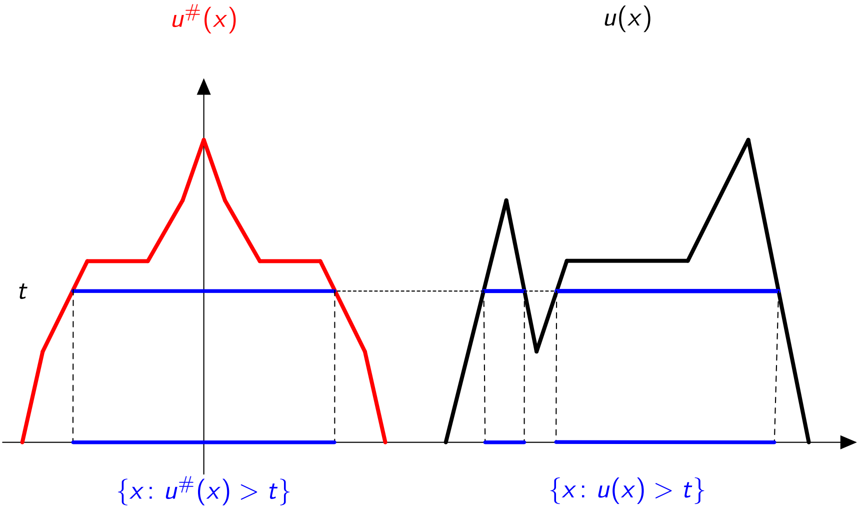

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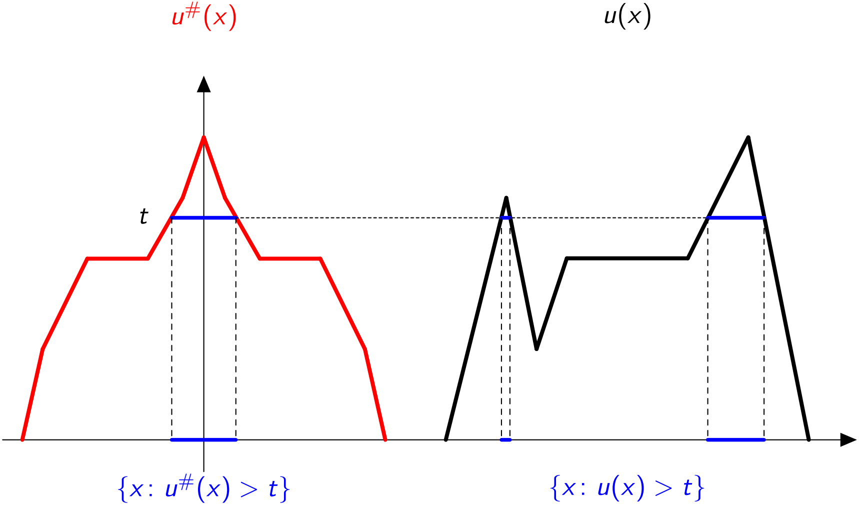

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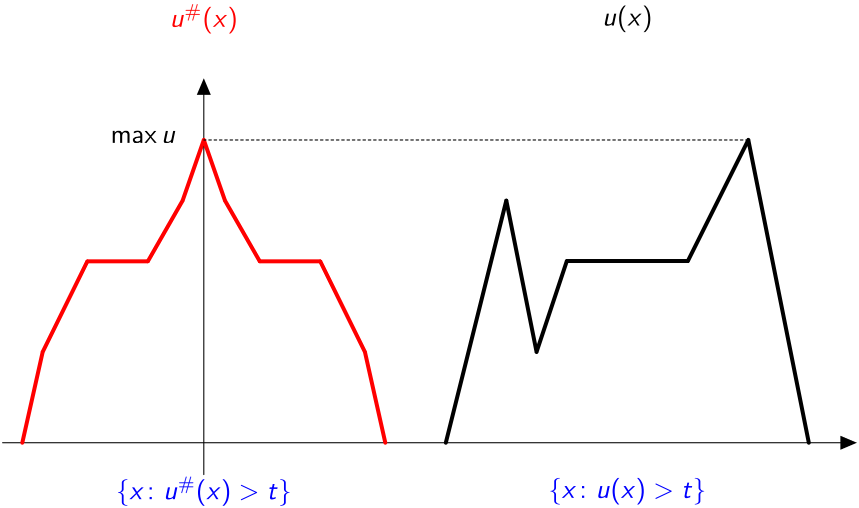
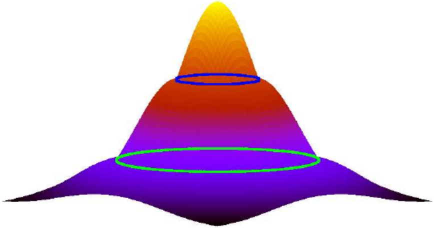


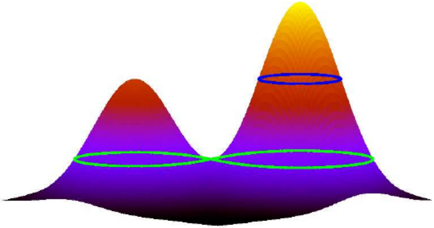
Symmetrization of functions

$u\colon \Omega \to \mathbb{R}$ measurable function, $\Omega \subset \mathbb{R}^n$, $0 < |\Omega| < \infty$

$\Omega^\#$ ball centered at the origin, $|\Omega^\#| = |\Omega|$



$u^\#$



u

Spherical decreasing rearrangement, $u^\#$

(1) $u^\#$ is radially symmetric decreasing

(2) $|\{x \in \Omega^\#\colon u^\#(x) > t\}| = |\{x \in \Omega\colon |u(x)| > t\}|$

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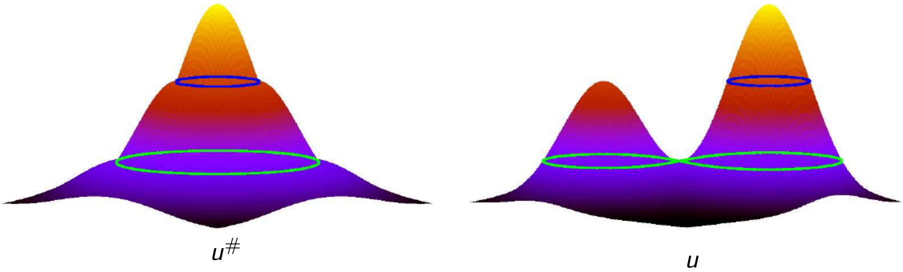

Cavalieri principle: the L^p norm is preserved

$$\int_\Omega |u|^p dx = \int_{\Omega^\#} |u^\#|^p dx, \qquad \forall p \geq 1$$

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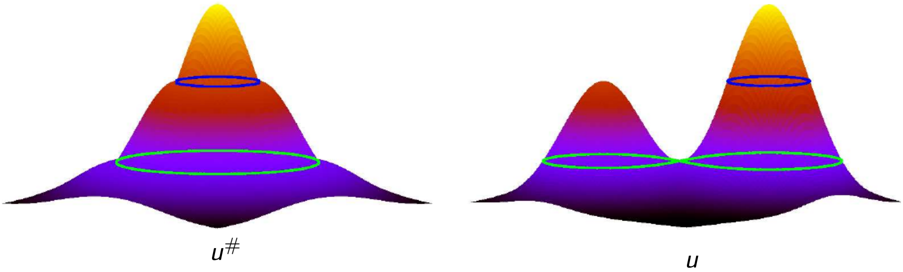

Pólya-Szegő: the energy decreases

If u has compact support, $\displaystyle\int_{\mathbb{R}^n} |\nabla u|^2\,dx \geq \int_{\mathbb{R}^n} |\nabla u^\#|^2\,dx$

Two classical problems

Partial differential equations: a classical result

$$\text{(P)} \qquad \begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{su } \partial\Omega \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^n$, and $\Delta u(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x)$.

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Among all the problems (P), for what data we have the "largest solution"?

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Maz'ja 1969, Talenti 1976

$$u^{\#}(x) \le v(x) \qquad \forall x \in \Omega^{\#}.$$

A consequence of Talenti's result

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Saint Venant conjecture

If $f \equiv 1$ then

$$T(\Omega) = \int_\Omega u\,dx = \int_{\Omega^\#} u^\# dx \le \int_{\Omega^\#} v\,dx = T(\Omega^\#).$$

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The torsional rigidity $T(\Omega)$ of an elastic bar with cross section Ω of fixed measure is maximal when Ω is a disk.

A classical shape optimization problem

Lord Rayleigh (1877)

Among all the elastic membranes, fixed at the boundary, with constant density and given area, the circular one has the lowest principal frequency
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Pólya-Szegő conjecture

Among all the $n-$sided polygonal membranes with constant density and given area, the regular one has the lowest principal frequency

Proved by Pólya and Szegő for $n = 3$ and $n = 4$, open for $n \geq 5$.

It holds:

$$\min_{|\Omega|=k} \lambda_1(\Omega) = \lambda_1(\Omega^{\#}), \qquad \sup_{|\Omega|=k} \lambda_1(\Omega) = +\infty.$$

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Pólya (1958)

If Ω is a bounded *convex* domain in $\mathbb{R}^2$, then

$$\lambda_1(\Omega) \le \frac{\pi^2}{4} \frac{Per(\Omega)^2}{|\Omega|^2}$$

and it is optimal for the slab.

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Payne - Weinberger (1961)

If Ω is a bounded *simply connected* domain in $\mathbb{R}^2$, then

$$\lambda_1(\Omega) \le \lambda_1^{ND}(A)$$

where A is a annulus with $|A| = |\Omega|$ and $Per(\Omega)$ is equal to the perimeter of the outer circumference, λ_1^{ND} has Dirichlet on the outer circumference and Neumann on the inner.

Hersch inequality (1960)

Let $\Omega \subset \mathbb{R}^2$ be a bounded *convex* domain. Then

$$\lambda_1(\Omega) \geq \left(\frac{\pi}{2}\right)^2 \frac{1}{\rho^2(\Omega)}$$

where $\rho(\Omega)$ is the inradius of Ω. The equality sign holds in the limiting case when Ω approaches a slab.

Key arguments

- Use of the so-called “web functions” (Pólya)

$$\varphi(x) = v(d(x, \partial\Omega)), \quad x \in \Omega.$$

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- on the level sets of u, we have

$$\Delta u = (N-1)H\, u_\nu + u_{\nu\nu}$$

Convex symmetrization

$$\lambda_{1,F}(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega F(\nabla u)^p dx}{\displaystyle\int_\Omega |u|^p \, dx}$$

with F given norm in $\mathbb{R}^N$.

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Efficiency of the first eigenfunction

Let Ω be a bounded domain in $\mathbb{R}^n$.

$$\lambda_1(\Omega) = \inf_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla \varphi|^2}{\displaystyle\int_\Omega \varphi^2}.$$

Let u_Ω be a positive eigenfunction relative to $\lambda_1(\Omega)$. The *efficiency* or *mean to max ratio* of u_Ω is defined by

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The functional $E(\Omega)$ is scaling invariant.

Efficiency of the first eigenfunction

Payne-Stakgold, 1973

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Chiti, 1982 (Reverse Hölder inequality)

If Ω is a bounded domain in $\mathbb{R}^N$, then

$$E(\Omega) \geq E(B)\frac{|B|}{|\Omega|},$$

where B is the ball in $\mathbb{R}^N$ such that $\lambda(B) = \lambda(\Omega)$. Equality holds if Ω is a ball.

Some examples

If $\triangle \subset \mathbb{R}^2$ is an equilateral triangle, then

$$E(\triangle) = \frac{2}{\pi\sqrt{3}}.$$

If $\square \subset \mathbb{R}^2$ is any rectangle, then

$$E(\square) = \frac{4}{\pi^2}.$$

If $B \subset \mathbb{R}^2$ is a disc, then

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Theorem (van den Berg, D.P., di Blasio, Gavitone, J. Spectral Theory, in press)

If $R > 0$, $\varepsilon > 0$, and $\Omega_{R,R+\varepsilon} = \{x \in \mathbb{R}^N \colon R < |x| < R + \varepsilon\}$, then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \lambda(\Omega_{R,R+\varepsilon}) = \pi^2,$$

$$\lim_{\varepsilon \downarrow 0} E(\Omega_{R,R+\varepsilon}) = \frac{2}{\pi}.$$

Lower bound

What about lower bounds on $E(\Omega)$?

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Theorem (van den Berg, D.P., di Blasio, Gavitone, J. Spectral Theory, in press)

There exists $C = C(N) > 0$ s.t. for all open, connected, bounded domains Ω in $\mathbb{R}^N$,

$$C\,\frac{\rho(\Omega)^N}{|\Omega|} \leq E(\Omega)$$

where $\rho(\Omega)$ is the inradius of Ω.
There exists $C > 0$ s.t. for all Ω open, planar, bounded, and convex domain,

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Theorem (van den Berg, D.P., di Blasio, Gavitone, J. Spectral Theory, in press)

It holds

$$\inf E(\Omega) = 0$$

where the infimum is computed among all the bounded convex sets Ω of $\mathbb{R}^N$.

Lower bound

We find (explicit) classes of convex domains Ω_n for which $E(\Omega_n) \to 0$. For example, in $\mathbb{R}^2$: shrinking circular sectors, rhombi, ellipses. We are able to give a decay rate on $E(\Omega_n)$ as $n \to +\infty$.

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The key point in our argument is given by a localizing property of the eigenfunctions:

Localising sequences

Let (Ω_n) be a sequence of non-empty bounded open sets in $\mathbb{R}^N$. We say that a sequence of first eigenfunctions $u_n = u_{\Omega_n}$ with $u_n \in L^2(\Omega_n)$, $n \in \mathbb{N}$ and $\|u_n\|_2 = 1$ localises if there exists a sequence of measurable sets $A_n \subset \Omega_n$ such that

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Proposition (van den Berg, D.P., di Blasio, Gavitone JST, in press)

If u_n localises, then Ω_n has vanishing efficiency.

Upper bound

By Payne-Stakgold, for convex domain in $\mathbb{R}^N$ it holds

$$E(\Omega) \leq \frac{2}{\pi}.$$

As we have seen, it is sharp for shrinking spherical shells. For a general domain, it holds that

$$E(\Omega) = \frac{1}{|\Omega| \max u} \int_\Omega u dx \leq 1.$$

Theorem (van den Berg, Bucur, Kappeler 2020)

$$\sup\{E(\Omega),\ \Omega \textit{ bounded, connected in } \mathbb{R}^N\} = 1.$$

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- is it the Payne-Stakgold sharp for convex domains of $\mathbb{R}^N$, or simply connected of $\mathbb{R}^2$?
- to find the best constants. Is it

$$E(\Omega) \leq E(B),$$

with B ball of $\mathbb{R}^N$ and Ω convex?

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Open problems

- is it the Payne-Stakgold sharp for convex domains of $\mathbb{R}^N$, or simply connected of $\mathbb{R}^2$?
- to find the best constants. Is it

$$E(\Omega) \leq E(B),$$

  with B ball of $\mathbb{R}^N$ and Ω convex?
- What about more general functionals involving $L^p - L^q$ norms of u_Ω, instead of $L^1 - L^\infty$?
  Kröger 1996: $\frac{\int_\Omega u_\Omega^2 dx}{|\Omega|(\max u_\Omega)^2}$.

Upper bound

Open problems

- is it the Payne-Stakgold sharp for convex domains of $\mathbb{R}^N$, or simply connected of $\mathbb{R}^2$?
- to find the best constants. Is it

$$E(\Omega) \leq E(B),$$

  with B ball of $\mathbb{R}^N$ and Ω convex?
- What about more general functionals involving $L^p - L^q$ norms of u_Ω, instead of $L^1 - L^\infty$?
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Thank you for your attention!