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Optimal control of the drift in Fokker-Planck equations

… with the drift in BV

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CAA-AvH mini workshop, April 30, 2021

Optimal control problem with a FP equation

$$\min_{(y,u)} \quad \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |y - y_d|^2 + \frac{\beta}{2}\int_0^T |u|^2$$

$$\text{s.t.} \quad \partial_t y + \operatorname{div}(\alpha u y) - \operatorname{div}(\sigma\sigma^\top \nabla y) = 0 \qquad \text{on } (0,T)\times\mathbb{R}^d,$$

$$y(0) = y_0 \qquad \text{on } \mathbb{R}^d$$

- (classical) FP equation $\leftrightarrow$ temporal evolution of *pdf*s $\leftrightarrow$ S(P)DE modeling
- temporal **control** u, **stationary** spatial drift vector α
- here: **very low regularity** compared to related work ([FG17; BKP18; AT21])
- physical motivation: nonsmooth potentials (bistable W form) ([Ris89; OD19])
- **disclaimer**: work in progress! today: **only** steps towards reduced gradient

Fokker-Planck equation

$$\partial_t y + \operatorname{div}(\alpha u y) - \operatorname{div}(\sigma\sigma^\top \nabla y) = 0 \qquad \text{on } (0,T) \times \mathbb{R}^d$$

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General assumptions

- **drift**: $b := u \otimes \alpha$ with $0 \le u \in L^2(0,T)$ and $\alpha \in BV_{loc} \cap L^{2+\delta}(\mathbb{R}^d;\mathbb{R}^d)$
- **divergence regularity**: $\operatorname{div}(\alpha) \in L^{1+\delta}(\mathbb{R}^d)$ and $\left[\operatorname{div}(\alpha)\right]^- \in L^\infty(\mathbb{R}^d)$
- **diffusion matrix**: $\sigma \in L^2(0,T;W^{1,2}_{loc}(\mathbb{R}^d))^{d\times m}$ (+ later: $\sigma\sigma^\top \succ 0$ uniformly)
- **initial value**: $y_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$

- analysis for transport/FP-equations ([DL89; Amb04; BL08; Luo12; BL19])

Fokker-Planck equation: existence & uniqueness

$$\partial_t y + \operatorname{div}(\alpha u y) - \operatorname{div}(\sigma\sigma^\top \nabla y) = 0 \qquad \text{on } (0,T) \times \mathbb{R}^d$$

$$y(0) = y_0 \qquad \text{on } \mathbb{R}^d$$

Theorem ([Amb04; BL08; Luo12])

For every $0 \le u \in L^2(0,T)$*: exists unique solution* $y = y_u$ *in the* ***weak sense****,*

$$y \in L^\infty(0,T; L^1 \cap L^\infty(\mathbb{R}^d)) \quad \textit{with} \quad \sigma^\top \nabla y \in L^2(0,T; L^2(\mathbb{R}^d))^m.$$

Moreover, y *is the* ***strongly renormalized solution****: for every* $\beta \in C^2(\mathbb{R})$*,*

$$\begin{aligned} &\partial_t \beta(y) + \operatorname{div}\big(\alpha u \beta(y)\big) - \operatorname{div}\big(\sigma\sigma^\top \nabla \beta(y)\big) \\ &\qquad - \big(\beta(y) - y\beta'(y)\big)\operatorname{div}(\alpha u) + \frac{1}{2}\beta''(y)\big|\sigma^\top \nabla y\big|^2 = 0 \quad \textit{in } \mathscr{D}'((0,T)\times\mathbb{R}^d). \end{aligned}$$

Inhomogeneous Fokker-Planck: existence (bounded)

$$\partial_t p + \operatorname{div}(bp) - \operatorname{div}(\sigma\sigma^\top \nabla p) = g \qquad \text{on } (0,\mathsf{T}) \times \mathbb{R}^d$$

$$p(0) = p_0 \qquad \text{on } \mathbb{R}^d$$

with

$$g \in \mathrm{L}^1(0,\mathsf{T};\mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^d)), \quad -\left[\operatorname{div}(b)\right]^- \leq c \in \mathrm{L}^1(0,\mathsf{T}).$$

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Obtain **some** solution:

- spatially mollify the data b, σ, p_0 $\longrightarrow$ smooth solutions p_ε to mollified
- maximum principle:

$$\|p_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le e^{\|c\|_{L^1(0,T)}} \left(\|p_0\|_{L^\infty(\mathbb{R}^d)} + \|g\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}\right)$$

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- spatially mollify the data b, σ, p_0 $\quad\longrightarrow\quad$ smooth solutions p_ε to mollified
- maximum principle & testing with regularized $|p_\varepsilon|$:

$$\|p_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \leq e^{\|c\|_{L^1(0,T)}}\left(\|p_0\|_{L^1(\mathbb{R}^d)} + \|g\|_{L^1(0,T;L^1(\mathbb{R}^d))}\right)$$

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- extract weak-$\star$ subsequence, pass to the limit solution p (keep the bounds!)

Inhomogeneous Fokker-Planck: uniqueness (bounded)

$$\partial_t p + \operatorname{div}(bp) - \operatorname{div}(\sigma\sigma^\top \nabla p) = g \qquad \text{on } (0,T) \times \mathbb{R}^d$$

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- idea: **renormalization** property gives uniqueness

$$\partial_t \beta(p) + \operatorname{div}\big(\alpha u \beta(p)\big) - \operatorname{div}\big(\sigma\sigma^\top \nabla \beta(p)\big)$$
$$- \big(\beta(p) - p\beta'(p)\big)\operatorname{div}(\alpha u) + \frac{1}{2}\beta''(p)\big|\sigma^\top \nabla y\big|^2 = 0 \quad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^d).$$

Inhomogeneous Fokker-Planck: uniqueness (bounded)

$$\begin{aligned} \partial_t p + \operatorname{div}(bp) - \operatorname{div}(\sigma\sigma^\top \nabla p) &= g && \text{on } (0,T) \times \mathbb{R}^d \\ p(0) &= p_0 && \text{on } \mathbb{R}^d \end{aligned}$$

- idea: **renormalization** property gives uniqueness $\longrightarrow$ "$\cdot\, \beta'(p)$" & chain rule

$$\partial_t p^2 + \operatorname{div}\big(\alpha u p^2\big) - \operatorname{div}\big(\sigma\sigma^\top \nabla p^2\big) + p^2 \operatorname{div}(\alpha u) \le 0 \quad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^d).$$

$$\downarrow$$

(distributional) Gronwall argument for $p^2 \quad \longrightarrow \quad p^2 \equiv 0 \quad \text{if } p_0 = 0$

Inhomogeneous Fokker-Planck: uniqueness (bounded)

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$$p(0) = p_0 \qquad \text{on } \mathbb{R}^d$$

- idea: **renormalization** property gives uniqueness $\longrightarrow$ "$\cdot\, \beta'(p)$" & chain rule
- **but**: equation does not hold pointwise $\longrightarrow$ regularize again ($p_\varepsilon := \rho_\varepsilon \star p$)

$$\rho_\varepsilon \star g = \rho_\varepsilon \star \Big[\partial_t p + \operatorname{div}(bp) - \operatorname{div}(\sigma\sigma^\top \nabla p)\Big]$$

$$= \partial_t p_\varepsilon + \operatorname{div}(bp_\varepsilon) - \operatorname{div}(\sigma\sigma^\top \nabla p_\varepsilon) + R_\varepsilon \quad \textbf{pointwise}$$

Inhomogeneous Fokker-Planck: uniqueness (bounded)

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$$= \partial_t p_\varepsilon + \operatorname{div}(bp_\varepsilon) - \operatorname{div}(\sigma\sigma^\top \nabla p_\varepsilon) + R_\varepsilon \quad \textbf{pointwise}$$

- $\beta'(p_\varepsilon)R_\varepsilon \to 0$ in $\mathscr{D}'((0,\mathsf{T}) \times \mathbb{R}^d)$ (**commutator lemma,** [DL89; Amb04])
- reverse chain rule $\longrightarrow$ renormalization property $\longrightarrow$ uniqueness!

Towards a reduced gradient

Formally: relevant linearized problem for $u \mapsto y_u \in C^1$, direction $h \in L^\infty(0,T)$:

$$\partial_t z + \operatorname{div}(\alpha \bar{u} z) - \operatorname{div}(\sigma\sigma^\top \nabla z) = -\operatorname{div}(\alpha h \bar{y}) \qquad \text{on } (0,T) \times \mathbb{R}^d$$

$$z(0) = 0 \qquad \text{on } \mathbb{R}^d$$

- distributional right-hand side

Under our assumptions: $\operatorname{div}(\alpha h \bar{y})$ **regular distribution**

$$\operatorname{div}(\alpha h \bar{y}) = h\alpha \cdot \nabla \bar{y} + h \operatorname{div}(\alpha)\bar{y} \in L^1(0,T;L^r(\mathbb{R}^d)) \quad (\text{some } 1 < r < 2)$$

$$\ldots \textbf{ if } \alpha \cdot \nabla \bar{y} \in L^1(0,T;L^r(\mathbb{R}^d)) \qquad (\text{e.g. if } \sigma\sigma^\top \succ 0 \quad \longrightarrow \quad \nabla y \in L^2(0,T;L^2(\mathbb{R}^d)))$$

- solution for FP with (spatially) unbounded right-hand side $L^1(0,T;L^r(\mathbb{R}^d))$?

Inhomogeneous Fokker-Planck (unbounded)

$$\partial_t z + \operatorname{div}(\alpha\bar{u}z) - \operatorname{div}(\sigma\sigma^\top\nabla z) = -\operatorname{div}(\alpha h\bar{y}) \qquad \text{on } (0,\mathrm{T})\times\mathbb{R}^d$$

$$z(0) = 0 \qquad \text{on } \mathbb{R}^d$$

No solution at all?

Recall *a priori* bound from mollified equation: $(1 < r < 2$ only$)$

$$\|z\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d))} \leq e^{\|c\|_{\mathrm{L}^1(0,\mathrm{T})}}\|\operatorname{div}(\alpha h\bar{y})\|_{\mathrm{L}^1(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d))}$$

- in general: $\alpha\bar{u}z \notin \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d))$ $\longrightarrow$ **cannot** pass to limit in equation!

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- in general: $\alpha \bar{u} z \notin L^1(0,T;L^1_{loc}(\mathbb{R}^d))$ $\longrightarrow$ **cannot** pass to limit in equation!

But: $\alpha \bar{u} \beta(z) \in L^1(0,T;L^1_{loc}(\mathbb{R}^d))$ when β bounded $\longrightarrow$ **renormalized** setup?

- approximate $\operatorname{div}(\alpha h \bar{y})$ by *bounded* data $g_k \in L^1(0,T;L^1 \cap L^\infty(\mathbb{R}^d))$
- pass to the limit $k \to \infty$ in **renormalized** form for solutions z_k

Inhomogeneous Fokker-Planck (unbounded)

$$\partial_t p + \operatorname{div}(bp) - \operatorname{div}(\sigma\sigma^\top \nabla p) = g \qquad \text{on } (0,\mathrm{T}) \times \mathbb{R}^d$$

$$p(0) = p_0 \qquad \text{on } \mathbb{R}^d$$

Theorem

For every $g \in \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d))$ and $p_0 \in \mathrm{L}^r(\mathbb{R}^d)$: exists unique solution

$$p \in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d)) \quad \text{with} \quad \sigma^\top \nabla p_r \in \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\mathbb{R}^d))^m$$

*in the **weakly renormalized sense**:*

$$\partial_t \beta(p) + \operatorname{div}\big(b\beta(p)\big) - \operatorname{div}\big(\sigma\sigma^\top \nabla \beta(p)\big) - \big(\beta(p) - p\beta'(p)\big)\operatorname{div}(b) + \frac{1}{2}\beta''(p)\left|\sigma^\top \nabla p\right|^2 = \beta'(p) g \quad \text{in } \mathscr{D}'((0,\mathrm{T}) \times \mathbb{R}^d)$$

*for every **bounded** $\beta \in \mathrm{C}^2(\mathbb{R})$ with **compactly supported derivative** β'. Moreover*

$$\|p\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d))} \leq e^{\|c\|_{\mathrm{L}^1(0,\mathrm{T})}} \big(\|p_0\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^1(0,\mathrm{T};\mathrm{L}^r(\mathbb{R}^d))}\big).$$

Towards a reduced gradient

$$\min_{u \in L^2(0,T)} \quad j(u) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |y_u - y_d|^2 + \frac{\beta}{2}\int_0^T |u|^2$$

$$\text{s.t.} \quad u \geq 0$$

A "manual" ansatz

$$j(\bar{u} + \varepsilon h) - j(\bar{u}) = \frac{1}{2}\|y_{\bar{u}+\varepsilon h} - y_{\bar{u}}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \int_0^T \int_{\mathbb{R}^d} (y_{\bar{u}+\varepsilon h} - y_{\bar{u}})(y_{\bar{u}} - y_d)$$

$$+ \varepsilon\beta \int_0^T \bar{u}h + \varepsilon^2 \frac{\beta}{2}\|h\|^2_{L^2(0,T)}$$

- divide by $\varepsilon \searrow 0 \quad \longrightarrow \quad$ first term vanish & second $\sim$ gradient?

Towards a reduced gradient: first term

$$\partial_t z_\varepsilon + \operatorname{div}(\alpha \bar{u} z_\varepsilon) - \operatorname{div}(\sigma\sigma^\top \nabla z_\varepsilon) = -\varepsilon \operatorname{div}(\alpha h \bar{y}_{\bar{u}+\varepsilon h}) \qquad \text{on } (0,\mathsf{T}) \times \mathbb{R}^d$$

$$z_\varepsilon(0) = 0 \qquad \text{on } \mathbb{R}^d$$

- $z_\varepsilon = y_{\bar{u}+\varepsilon h} - y_{\bar{u}} \in \mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^d))$ is the **unique** (weak) solution
- right-hand side regular distribution in $\mathrm{L}^1(0,\mathsf{T};\mathrm{L}^r(\mathbb{R}^d))$

$$\|z_\varepsilon\|_{\mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^r(\mathbb{R}^d))} \lesssim \varepsilon,$$

$$\|z_\varepsilon\|_{\mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^\infty(\mathbb{R}^d))} \lesssim 1$$

Interpolation: $\left(\frac{1}{2} = \frac{r}{2} \cdot \frac{1}{r} + \frac{2-r}{2} \cdot \frac{1}{\infty}\right)$

$$\|y_{\bar{u}+\varepsilon h} - y_{\bar{u}}\|^2_{\mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^2(\mathbb{R}^d))} = \|z_\varepsilon\|^2_{\mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^2(\mathbb{R}^d))} \lesssim \varepsilon^r, \quad r > 1$$

Towards a reduced gradient: second term

Adjoint equation: (**renormalized** notion of solution as before)

$$-\partial_t q - \alpha \bar{u} \cdot \nabla q - \operatorname{div}(\sigma\sigma^\top \nabla q) = y_{\bar{u}} - y_d \qquad \text{in } (0, \mathsf{T}) \times \mathbb{R}^d,$$

$$q(\mathsf{T}) = 0 \qquad \text{on } \mathbb{R}^d.$$

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Theorem (Duality formula)

$$\int_0^{\mathsf{T}} \int_{\mathbb{R}^d} (y_{\bar{u}+\varepsilon h} - y_{\bar{u}})(y_{\bar{u}} - y_d) = \varepsilon \int_0^T h \int_{\mathbb{R}^d} y_{\bar{u}+\varepsilon h}\, \alpha \cdot \nabla q$$

Ansatz: test adjoint equation with z_ε, “z_ε equation” with q, integrate by parts

- adjoint equation in renormalized form
- mollify both equations to have regular solutions, get rid of all error terms

Reduced gradient

$$\min_{u \in L^2(0,T)} \quad j(u) = \int_0^T \int_{\mathbb{R}^d} |y_u - y_d|^2 + \frac{\beta}{2} \int_0^T |u|^2$$

$$\text{s.t.} \quad u \geq 0$$

The "manual" ansatz

$$j(\bar{u} + \varepsilon h) - j(\bar{u}) = \frac{1}{2} \|y_{\bar{u}+\varepsilon h} - y_{\bar{u}}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \int_0^T \int_{\mathbb{R}^d} \left(y_{\bar{u}+\varepsilon h} - y_{\bar{u}}\right)\left(y_{\bar{u}} - y_d\right)$$

$$+ \varepsilon\beta \int_0^T \bar{u}h + \varepsilon^2 \frac{\beta}{2} \|h\|^2_{L^2(0,T)}$$

Reduced gradient

$$\min_{u \in \mathrm{L}^2(0,\mathrm{T})} \quad j(u) = \int_0^{\mathrm{T}} \int_{\mathbb{R}^d} |y_u - y_d|^2 + \frac{\beta}{2} \int_0^{\mathrm{T}} |u|^2$$

$$\text{s.t.} \quad u \geq 0$$

The “manual” ansatz

$$j(\bar{u} + \varepsilon h) - j(\bar{u}) \simeq \varepsilon^r + \varepsilon \int_0^T h \int_{\mathbb{R}^d} y_{\bar{u}+\varepsilon h}\, \alpha \cdot \nabla q$$

$$+ \varepsilon\beta \int_0^{\mathrm{T}} \bar{u}h + \varepsilon^2 \frac{\beta}{2} \|h\|^2_{\mathrm{L}^2(0,\mathrm{T})}$$

Reduced gradient

$$\min_{u\in L^2(0,T)} \quad j(u) = \int_0^T \int_{\mathbb{R}^d} |y_u - y_d|^2 + \frac{\beta}{2}\int_0^T |u|^2$$

$$\text{s.t.} \quad u \geq 0$$

Reduced gradient in $L^2(0,T)$

$$\nabla j(\bar{u}) = \int_{\mathbb{R}^d} y_{\bar{u}}\, \alpha \cdot \nabla q + \beta \bar{u}$$

$$-\partial_t q - \alpha \bar{u} \cdot \nabla q - \operatorname{div}(\sigma\sigma^\top \nabla q) = y_{\bar{u}} - y_d \qquad \text{in } (0,T)\times\mathbb{R}^d,$$

$$q(T) = 0 \qquad \text{on } \mathbb{R}^d.$$

Wrap-up & outlook

Seen today:

- Fokker-Planck equation with irregular data
- strongly & weakly renormalized solutions for bounded & unbounded data
- assumptions such that $\operatorname{div}(\alpha h y_{\bar{u}})$ regular distribution ($\sigma\sigma^\top \succ 0$ needed?)
- no IFT framework, but obtain reduced gradient

Omitted today:

- stability results, (weak) continuity properties $u \mapsto y_u$ (used implicitly at end)
- a lot of details

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Do not know yet:

- any chance for second order approximations? ($\alpha \in \mathrm{L}^\infty(\mathbb{R}^d)$?)
- circumvent $\sigma\sigma^\top \succ 0$ assumption? (Vanishing viscosity, transport equation?)
- some details

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Notes