Control for heterogeneous 1D reaction-diffusion equations

Maicon Sônego

Federal University of Itajubá-MG, Brazil
Visiting researcher at FAU

Joint work with Enrique Zuazua

For $L > 0$, $0 \leq T \leq +\infty$, consider the following controlled reaction-diffusion equation on $(0, L) \times (0, T)$

$$\begin{cases} y_t = (a(x)y_x)_x + b(x)f(y), \\ y(0,t) = u(t), \quad y(L,t) = v(t), \\ y(x,0) = y_0(x), \end{cases} \tag{1}$$

where f is *monostable* ($f(u) = u - u^2$, for instance) or *bistable* ($f(u) = u(u-1)(\theta - u)$, $\theta \in (0,1)$, for instance);
$a, b : (0, L) \to \mathbb{R}$ are positive functions of class C^2 and the controls u and v are measurable functions satisfying the constraints

$$0 \leq u(t) \leq 1, \quad 0 \leq v(t) \leq 1.$$

Let z be a steady state solution of (1) such that $0 \leq z \leq 1$. We say that the controlled equation (1) is

- *controllable in finite time towards z* if for any initial condition $0 \leq y_0 \leq 1$ in $L^\infty(0, L)$, there exists $0 \leq T < \infty$ and controls $u, v \in L^\infty(0, T; [0, 1])$ such that

$$y(T, \cdot) = z(\cdot).$$

- *controllable in infinite time towards z* if for any initial condition $0 \leq y_0 \leq 1$ in $L^\infty(0, L)$, there exists controls $u, v \in L^\infty(0, \infty; [0, 1])$ such that

$$y(t, \cdot) \to z(\cdot)$$

uniformly in $[0, L]$ as $t \to \infty$.

Some recent references

1. C. Pouchol, E. Trélat and E. Zuazua, *Phase portrait control for 1D monostable and bistable reaction–diffusion equations*, Nonlinearity, 2019.
2. D. Ruiz-Balet, E. Zuazua, *Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations*, Journal de Mathématiques Pures et Appliquées, 2020.
3. I. Mazari, D. Ruiz-Balet and E. Zuazua, *Constrained control of gene-flow models*, arXiv:2005.09236v4.

The main result

Our strategy will be to use the Matano's results on the asymptotic behavior of solutions of semilinear problems.

First, we present a very general result of existence and uniqueness.
Let $g(x, y, y')$ be a C^1 function such that

$$G_1(y-z, y'-z') \leq g(x, y, y') - g(x, z, z') \leq G_2(y-z, y'-z') \tag{2}$$

where

$$G_1(y, y') = \begin{cases} M_1 y' + K_1 y, & y \geq 0, \quad y' \geq 0 \\ M_2 y' + K_1 y, & y \geq 0, \quad y' \leq 0 \\ M_2 y' + K_2 y, & y \leq 0, \quad y' \leq 0 \\ M_1 y' + K_2 y, & y \leq 0, \quad y' \geq 0 \end{cases} \tag{3}$$

$$G_2(y,y') = \begin{cases} M_2 y' + K_2 y, & y \geq 0, \quad y' \geq 0 \\ M_1 y' + K_2 y, & y \geq 0, \quad y' \leq 0 \\ M_1 y' + K_1 y, & y \leq 0, \quad y' \leq 0 \\ M_2 y' + K_1 y, & y \leq 0, \quad y' \geq 0 \end{cases} \tag{4}$$

and $L_i, K_i \in \mathbb{R}$ $(i = 1, 2)$ are constant.

Theorem 1 (existence and uniqueness)

P. Bailey, L. F. Shampine, P. Waltman, 1966

For $(x, y, y') \in [0, L] \times [0, 1] \times \mathbb{R}$, let $g(x, y, y')$ be a continuous functions and satisfying (2). If the two problems

$$\begin{cases} u_i'' + G_i(u_i(x), u_i'(x)) = 0, \\ u_i(a) = \bar{A}, \quad u_i(b) = \bar{B} \end{cases} \tag{5}$$

have unique solutions on every interval $[a, b]$ of $[0, L]$ for arbitrary $\bar{A}, \bar{B}$, and if for $a = 0$, $b = L$, $\bar{A} = A$, $\bar{B} = B$ the ranges are subsets of $[0, 1]$, then the problem

$$\begin{cases} u'' + g(x, u(x), u'(x)) = 0, \\ u(0) = A, \quad u(L) = B \end{cases} \tag{6}$$

has a unique solution $u(x)$ which remains in $[0, 1]$.

In our case the steady states solutions satisfy:

$$\begin{cases} y_{xx} + \dfrac{a_x(x)}{a(x)} y_x + \dfrac{b(x)}{a(x)} f(y) = 0, \quad x \in (0, L) \\ y(0) = \bar{u}, \quad y(L) = \bar{v}. \end{cases} \tag{7}$$

Thus, in this case

$$g(x, u(x), u'(x)) = \frac{a'(x)}{a(x)} u'(x) + \frac{b(x)}{a(x)} f(u(x)).$$

We denote,

$$K^+ = \sup_{(u,v)\in[0,1]\times[0,1]} \left\{ \frac{f(u) - f(v)}{u - v} \right\}$$

and

$$K^- = \inf_{(u,v)\in[0,1]\times[0,1]} \left\{ \frac{f(u) - f(v)}{u - v} \right\}.$$

Hence, for any $(x, y, y'), (x, z, z') \in [0, L] \times [0, 1] \times \mathbb{R}$,

$$G_1(y - z, y' - z') \le g(x, y, y') - g(x, z, z') \le G_2(y - z, y' - z'),$$

where G_1 and G_2 are defined as (3) and (4), respectively, with

$$M_1 = \inf_{x \in [0,L]} \left\{ \frac{a'(x)}{a(x)} \right\}, \tag{8}$$

$$M_2 = \sup_{x \in [0,L]} \left\{ \frac{a'(x)}{a(x)} \right\}, \tag{9}$$

$$K_1 = \inf_{x \in [0,L]} \left\{ \frac{b(x)}{a(x)} K^- \right\} \tag{10}$$

and

$$K_2 = \sup_{x \in [0,L]} \left\{ \frac{b(x)}{a(x)} K^+ \right\}. \tag{11}$$

In order to state our main result, we define

$$\alpha(M,K) =$$

$$\begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0 \\[2ex] \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0, M > 0, K > 0 \\[2ex] \dfrac{2}{M}, & \text{if } 4K - M^2 = 0, M > 0 \\[2ex] +\infty, & \text{otherwise} \end{cases}$$

$$\beta(M, K) =$$

$$\begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0 \\[2ex] \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0,\ M < 0,\ K > 0 \\[2ex] \dfrac{-2}{M}, & \text{if } 4K - M^2 = 0,\ M < 0 \\[2ex] +\infty, & \text{otherwise} \end{cases}$$

Theorem 2

Consider $\bar{y}$ a steady state of (1) such that $\bar{y}(0) = \bar{u}$, $\bar{y}(L) = \bar{v}$ and $0 \leq \bar{u}, \bar{v} \leq 1$. If

$$L < \alpha(M_2, K_2) + \beta(M_1, K_2) \tag{12}$$

and the problems $(i = 1, 2)$

$$\begin{cases} z'' + G_i(z, z') = 0, \quad (0, L) \\ z(0) = \bar{u}, \quad z(L) = \bar{v} \end{cases} \tag{13}$$

have solutions with ranges contained in $[0, 1]$, then (1) is controllable in infinite time towards $\bar{y}$.

Example 1

Consider $a \equiv b \equiv 1$ and $f(u) = u(1-u)$. In this case, we have

$$\begin{cases} y_t = y'' + y(1-y), \quad (0, L) \\ y(0,t) = u(t), \quad y(L,t) = v(t), \\ y(x,0) = y_0(x). \end{cases} \tag{14}$$

Thus, we obtain

$$K^- = -1, \quad K^+ = 1$$

and then

$$M_1 = M_2 = 0, \quad K_1 = -1, \quad K_2 = 1.$$

Moreover,

$$\alpha(M_2, K_2) + \beta(M_1, K_2) =$$

$$\frac{2}{\sqrt{4K_2 - M_2^2}} \cos^{-1}\left(\frac{M_2}{2\sqrt{K_2}}\right) + \frac{2}{\sqrt{4K_2 - M_1^2}} \cos^{-1}\left(\frac{-M_1}{2\sqrt{K_2}}\right) =$$

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Hence, we take $L < \pi$ and if we want to analyse the solution $\bar{y} \equiv 0$ we can take controls $\bar{u} = \bar{v} = 0$ and the problems

$$\begin{cases} z'' + z = 0, & (0, L) \\ z(0) = 0, & z(L) = 0 \end{cases} \quad \text{and} \quad \begin{cases} z'' - z = 0, & (0, L) \\ z(0) = 0, & z(L) = 0. \end{cases}$$

It is easy to see that both have solution $z \equiv 0$ and, in these conditions, we can conclude that the problem (14) is controllable in infinite time towards $\bar{y} \equiv 0$.

Example 2

Now, we consider $a(x) = e^{5x}$, $b(x) = x + 6$ and again $f(u) = u(1 - u)$. Then, we have

$$\begin{cases} y_t = (e^{5x} y_x)_x + (x+6)y(1-y), & (0, L) \\ y(0,t) = u(t), \quad y(L,t) = v(t), \\ y(x,0) = y_0(x). \end{cases} \tag{15}$$

In this case we obtain,

$$M_1 = M_2 = 5, \quad K_1 = -6, \quad K_2 = 6,$$

and a simple computation give us

$$\alpha(M_2, K_2) + \beta(M_1, K_2) = \infty.$$

Now, we can, for example, analyze a target $\bar{y}$ such that $\bar{y}(0) = 1/2$ and $\bar{y}(L) = 1/4$. For this, and again adopting the strategy of static controls, we have to look for solutions of

$$\begin{cases} z'' + 5z' - 6z = 0, & (0, L) \\ z(0) = 1/2, & z(L) = 1/4 \end{cases} \quad \text{and} \quad \begin{cases} z'' + 5z' + 6z = 0, & (0, L) \\ z(0) = 1/2, & z(L) = 1/4. \end{cases}$$

So, if we take $L = 1$ for instance, we have the following solutions

$$z_1(x) = \frac{e^{6-6x} - 2e^{7-6x} + 2e^{x} - e^{6+x}}{4 - 4e^{7}}$$

and

$$z_2(x) = \frac{e^{-3x}(2e - e^{3} - 2e^{x} + e^{3+x}))}{4(-1 + e)}$$

Both solutions have ranges contained in $[0, 1]$ and then we can use our main result to conclude that (15) is controllable in infinite time towards $\bar{y}$.

Some heterogeneous non-linearities that could be considered with the proposed method

1. $f(u,x) = u(u-\theta(x))(1-u)$, $0 < \theta(x) < 1$ (related to *Fife-Greenlee equation*);

2. $f(u,x) = u(a^2(x) - u^2(x))$, $0 < a(x) < 1$;

3. $f(u,x) = \rho(x)u(1-u)$ (related to *Fisher-KPP equation*).

Thank you!

Maicon Sônego

mcn.sonego@unifei.edu.br