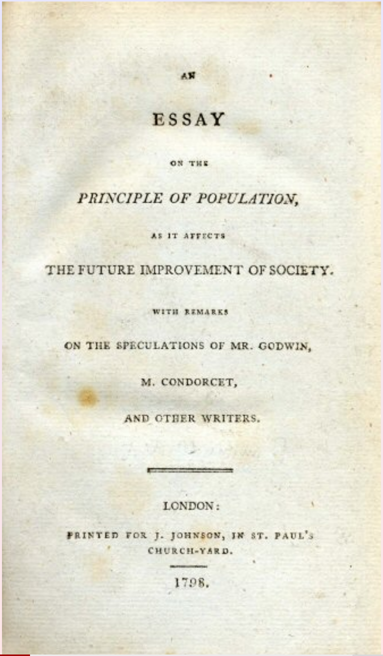
Control of a Population Dynamics Model with Age Structuring and Diffusion

Enrique Zuazua

FAU - AvH

April 3, 2020

Table of Contents



AN

ESSAY

ON THE

PRINCIPLE OF POPULATION,

AS IT AFFECTS

THE FUTURE IMPROVEMENT OF SOCIETY.

WITH REMARKS

ON THE SPECULATIONS OF MR. GODWIN,

M. CONDORCET,

AND OTHER WRITERS.

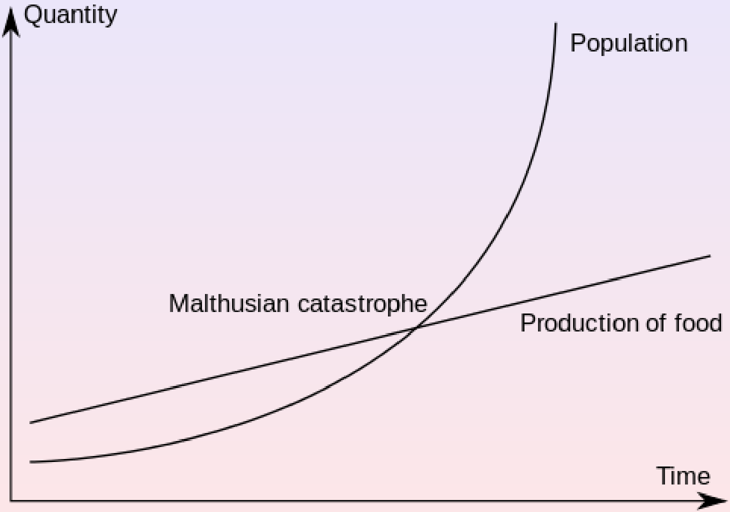
LONDON:

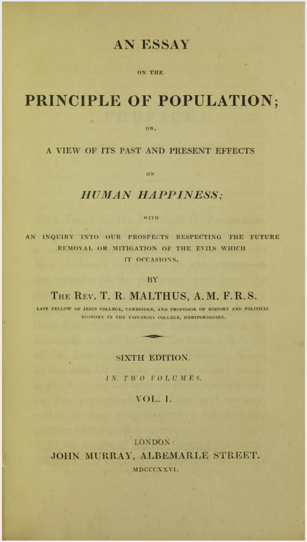
PRINTED FOR J. JOHNSON, IN ST. PAUL'S CHURCH-YARD.

1798.

- The book "An Essay on the Principle of Population" was first published **anonymously** in 1798, but the author was soon identified as Thomas Robert Malthus (1766-1834).
- It predicted that population would increase geometrically, doubling every 25 years, but food production would only grow arithmetically, which would result in famine and starvation, unless births were controlled (the **Malthusian trap**).
- Malthus' views became influential, and controversial, across economic, political, social and scientific thought.

Malthus estimated population collapse to occur in 1880





AN ESSAY

ON THE

PRINCIPLE OF POPULATION;

OR,

A VIEW OF ITS PAST AND PRESENT EFFECTS

ON

HUMAN HAPPINESS;

WITH

AN INQUIRY INTO OUR PROSPECTS RESPECTING THE FUTURE REMOVAL OR MITIGATION OF THE EVILS WHICH IT OCCASIONS.

BY

THE REV. T. R. MALTHUS, A.M. F.R.S.

LATE FELLOW OF JESUS COLLEGE, CAMBRIDGE, AND PROFESSOR OF HISTORY AND POLITICAL ECONOMY IN THE EAST-INDIA COLLEGE, HERTFORDSHIRE.

SIXTH EDITION.

IN TWO VOLUMES.

VOL. I.

LONDON:
JOHN MURRAY, ALBEMARLE STREET.
MDCCCXXVI.

The book's 6th edition (1826) was independently cited as a key influence by both Charles Darwin and Alfred Russel Wallace in developing the **theory of natural selection**.

Was Malthus right?

Some economists contend that **since the industrial revolution**, mankind has broken out of the trap.

But...

"A finite world can support only a finite population".

"The Tragedy of the Commons," Garrett Hardin, ecologist, 1968.

Nowadays Population Dynamics is a fertile area in Mathematical Biology, Social Sciences,...

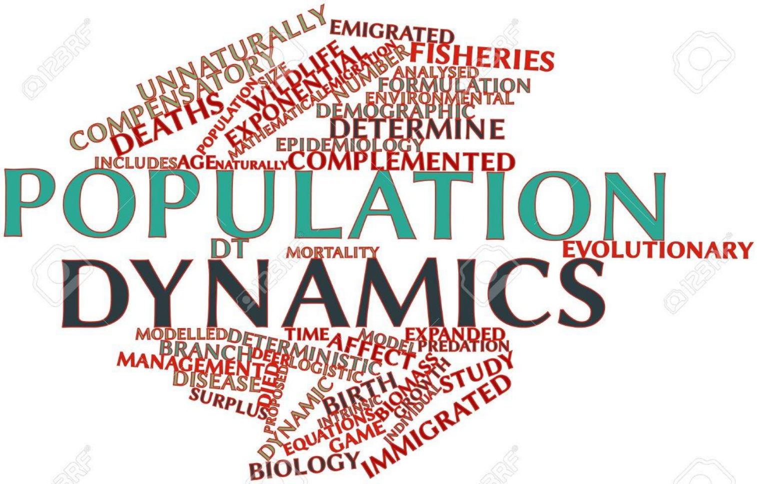


Table of Contents

A fundamental model

Morton E. Gurtin, A System of Equations for Age-dependent Population Diffusion, J. theor. Biol. (1973), 40, 389-392. (Letters to the Editor).

$$\rho_t + \rho_a - \Delta\rho + \lambda\rho = 0$$

with

$$\rho = \rho(x, a, t) = \text{population density}$$

$$a = \text{age}$$

$$\lambda = \lambda(a) = a = \text{death rate at age } a$$

Operator splitting

Hyperbolic-parabolic mix

$$\rho_t + \rho_a - \Delta_x \rho = 0$$

$$=$$

$$\rho_t + \rho_a = 0$$

$$+$$

$$\rho_t - \Delta_x \rho = 0$$

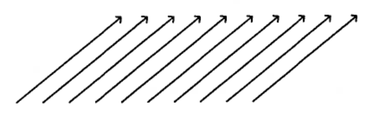
A second hidden parabolic equation as the generator of the semigroup

$$\rho_a - \Delta_x \rho = 0.$$

Age-time transport

$$\rho_t + \rho_a = 0$$

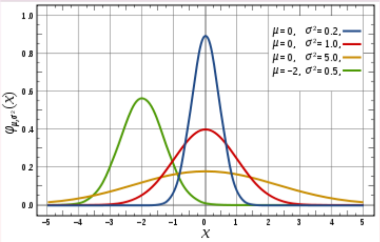
$$\rho = \rho_0(a - t)$$



Space-time diffusion

$$\rho_t - \Delta_x \rho = 0$$

$$\rho(\cdot, t) = G(\cdot, t) *_x \rho_0(\cdot)$$
$$G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t).$$



Overall

$$\rho_t + \rho_a - \Delta_x \rho = 0$$

$$\rho(x, a, t) = G(\cdot, t) *_x \rho_0(\cdot, a - t)$$

Smoothing in the space variable x but lack of regularising effect in the age variable a.
Lack of hypocoercivity/hypoellipticity due to commutativity of the two generators ∂_a and $-\Delta_x$, in contrast with the classical Kolmogorov equation:

$$f_t - f_{vv} + vf_x = 0.$$

Semigroup generation

W. Huyer, Semigroup formulation and approximation of a linear age-dependent population problem with spatial diffusion, Semigroup Forum, 49 (1994) 99–114.

With suitable boundary conditions it generates a semigroup in $H = L^2((0, a_\dagger) \times \Omega)$, Ω being the bounded domain of $\mathbf{R}^n$ for the space variable x, and $a_\dagger > 0$ the maximal age of the population:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta_x \rho + \mu(a)\rho = 0 & \text{in } (0,\infty) \times (0, a_\dagger) \times \Omega, \\ \rho(x,0,t) = \int_0^{a_\dagger} \beta(a)\rho(x,a,t)\, da & (t,x) \in (0,\infty) \times \Omega \\ \frac{\partial \rho}{\partial \nu} = 0 & (t,a,x) \in (0,\infty) \times (0,a_\dagger) \times \partial\Omega, \\ \rho(x,a,0) = \rho_0(x,a) & (a,x) \in (0,a_\dagger) \times \Omega. \end{cases}$$

Fertility and mortality rates β and μ are such that:

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta \geqslant 0$ for almost every $a \in (0, a_\dagger)$.

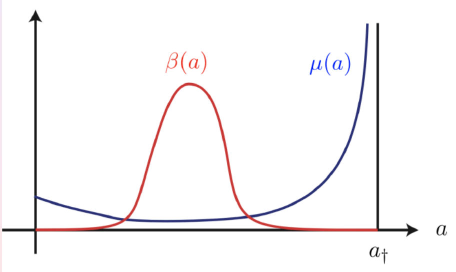
(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu \geqslant 0$ $a.e.$

(H3) $\int_0^{a_\dagger} \mu(a)\, da = +\infty$ (vanishing of the survival probability at the maximal age).

Nonlocal model

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta_x \rho + \mu(a)\rho = 0$$

$$\rho(x, 0, t) = \int_0^{a_\dagger} \beta(a)\rho(x, a, t)\, da.$$



Mortality-Fertility profiles

Dissipativity

• L^1

$$\frac{d}{dt}\left[\int_0^{a_\dagger}\int_\Omega \rho(x,a,t)dxda\right]$$

$$= -\int_0^{a_\dagger}\int_\Omega \mu(a)\rho(x,a,t)dxda + \int_0^{a_\dagger}\int_\Omega \beta(a)\rho(x,a,t)\,dxda.$$

• L^2

$$\frac{d}{dt}\left[\int_0^{a_\dagger}\int_\Omega \rho^2(x,a,t)dxda\right] = -\int_0^{a_\dagger}\int_\Omega |\nabla_x\rho(x,a,t)|^2dxda$$

$$-\int_0^{a_\dagger}\int_\Omega \mu(a)|\rho(x,a,t)|^2dxda + \int_\Omega \left|\int_0^{a_\dagger}\beta(a)\rho(x,a,t)\,da\right|^2dx.$$

Table of Contents

The control problem

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta_x \rho + \mu(a)\rho = u(x,a,t)\chi_{(a_1,a_2)\times\omega} & \text{in } \Omega \times (0,a_\dagger) \times (0,\infty) \\ \rho(x,0,t) = \int_0^{a_\dagger} \beta(a)\rho(x,a,t)\, da & \text{in } \Omega \times (0,\infty) \\ \frac{\partial \rho}{\partial \nu} = 0 & \text{in } (0,\infty) \times \partial\Omega \times (0,a_\dagger) \\ \rho(x,a,0) = \rho_0(x,a) & \text{in } \Omega \times (0,a_\dagger). \end{cases}$$

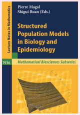
Here:

1. $u(x,a,t)$ is the control;
2. $(a_1,a_2) \times \omega \subset (0,a_\dagger) \times \Omega$ is the support of the control
3. Goal : To drive the solution to equilibrium at a given final time $T > 0$

$$\rho(\cdot,\cdot,T) \equiv 0.$$

Some references related to control

1. B. Ainseba and S. Aniţa, Internal exact controllability of the linear population dynamics with diffusion, Electron. JDE, (2004).
2. V. Barbu, M. Iannelli, and M. Martcheva, On the controllability of the Lotka-McKendrick model of population dynamics, JMAA, (2001).
3. N. Hegoburu, P. Magal, and M. Tucsnak, Controllability with positivity constraints of the Lotka- McKendrick system. 2017.
4. P. Martinez, J.-P. Raymond, and J. Vancostenoble, Regional null controllability of a linearized Crocco-type equation, SICON, (2003).
5. O. Traore, Null controllability of a nonlinear population dynamics problem, Int. J. Math. Math. Sci., (2006).
6. G. F. Webb, Theory of nonlinear age-dependent population dynamics, vol. 89 of Monographs and Text- books in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1985.



The dual observation problem

Consider the adjoint system

$$\begin{cases} \frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q - \beta(a)q(x,0,t) + \mu(a)q = 0, & \Omega \times (0, a_\dagger) \times (0, T) \\ q(t, a_\dagger, x) = 0, & \Omega \times (0, T) \\ \partial q / \partial \nu = 0, & \partial\Omega \times (0, a_\dagger) \times (0, T) \\ q(x, a, 0) = q_0(x, a), & \Omega \times (0, a_\dagger). \end{cases}$$

The question is whether:

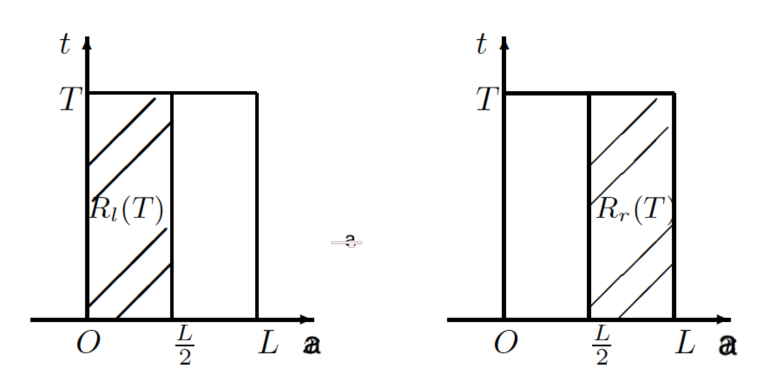
$$\int_0^{a_\dagger} \int_\Omega q^2(x, a, T)\, dx\, da \leqslant C_T \int_0^T \int_{a_1}^{a_2} \int_\omega q^2(x, a, t)\, dx\, da\, dt,$$

the observation being made in the subset

$$(a_1, a_2) \times \omega \subset (0, a_\dagger) \times \Omega.$$

Hyperbolic control

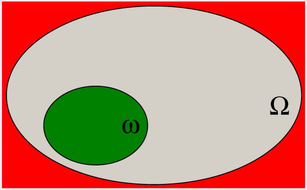
A hyperbolic system is controllable/observable if and only if the control is effective in each characteristic of the system[1]



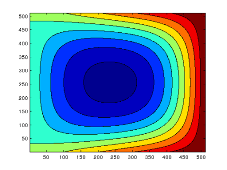
[1] Generalised to second order wave equations by Bardos, Lebeau and Rauch in 1988 through the so-called *Geometric Control Condition (GCC)*

Parabolic control

Heat-like equations are controllable from any open subset of the space subdomain, thanks to the infinite velocity of propagation, and the radial nature of diffusion.



Actual proofs use Carleman inequalities developed by Fursikov-Imanuvilov and Lebeau-Robbiano, among others, in the 90's.



The main result: Observability/controllability

Theorem. Controllability/observability holds with controls/observations in

$$(a_1, a_2) \times \omega \subset (0, a_\dagger) \times \Omega$$

provided the fertility profile $\beta(a)$ is such that

$$\beta(a) \equiv 0, \text{ in } (0, a_1 + \delta)$$

and the time T is large enough such that

$$T > a_1 + (a_\dagger - a_2).$$

Furthermore, when the initial state and target are positive, the trajectory may be kept non-negative if the control time T is large enough.

Proof. Observability

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q - \beta(a)q(x,0,t) + \mu(a)q = 0$$

Step 1. Since

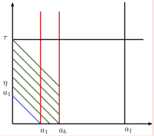
$$\beta(a) \equiv 0, \text{ in } (0, a_1 + \delta)$$

we can use the simplified equation

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu(a)q = 0$$

and deduce that

$$\int_{a_1}^{T} \int_{\omega} |q(x,0,t)|^2 dxdt \le C_T \int_0^T \int_{a_1}^{a_2} \int_{\omega} q^2(x,a,t)\, dx\, da\, dt.$$



Proof. Step 2. The case $\beta \equiv 0$.

For

$$t > a_1$$

the term in the equation involving the fertility profile β has been estimated and can be ignored. One is left with the reduced equation

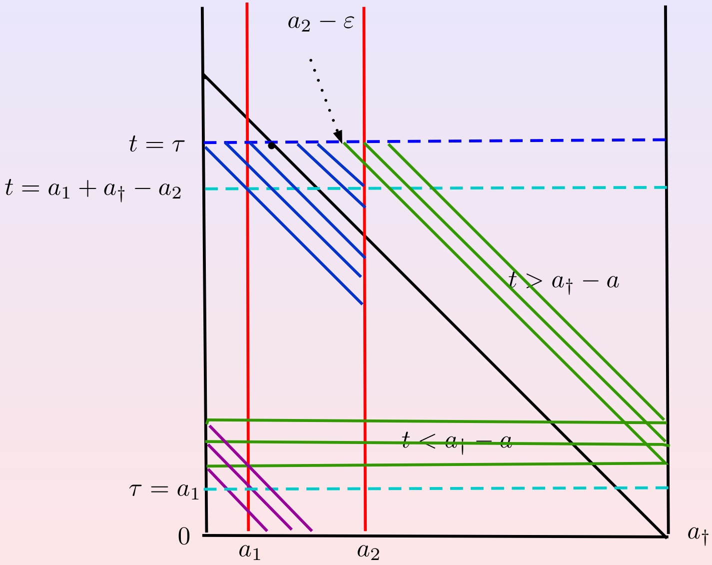
$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu(a) q = 0.$$

We start again at $t = t_1$.In this second phase observability requires the extra characteristic time:

$$T > a_\dagger - a_2.$$

This requires propagation along characteristics and the well-known estimates for the control/observation of the heat equation.

Graphical proof



Change of variables

Equation

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu(a)q = 0$$

is transformed into

$$\frac{\partial \eta}{\partial s} - \Delta \eta + \mu(b-s)\eta = 0$$

by the change of variables

$$\eta(x,s) = q(x, b-s, s)$$

where the parameter b is chosen according to the inequality under consideration.

This allows applying the existing observability inequalities for heat-like equations to our model.

Table of Contents

Positivity of controlled trajectories

In the context of population dynamics, the state ρ refers to a **density of population**.

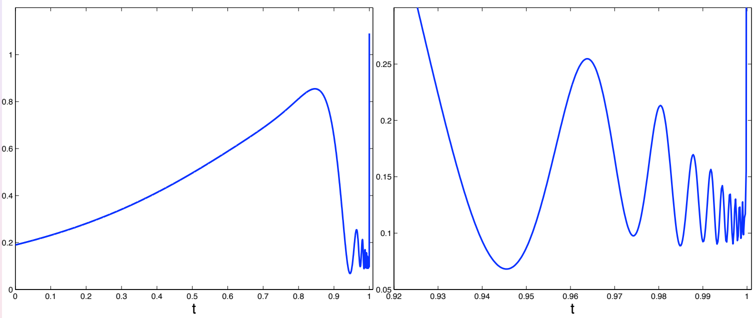
Thus, in principle, even if some slight overshoots can be admissible and even have some physical interpretation, one expects the trajectories to be always non-negative.

The following questions arises then:

If the initial state and the target are positive sates, can the controlled trajectory be guaranteed to be non-negative all along the control time horizon?

$$\rho \geq 0?$$

Optimal L^2 controls for heat equations do not fulfill natural constraints: Numerical simulations, back to Glowinski and Lions in the 80's and 90's



This is so since controls are just restrictions to ω of solutions of the adjoint system

$$-\varphi_t - \Delta\varphi = 0.$$

Bad news

1. Controls oscillate dramatically as time approaches the final time.
2. This produces oscillations in the state too.
3. This effect is further accentuated when the time horizon T is short, as $T \to 0$.
4. This makes the controllability results of little use in many contexts in which the state represents a density (population dynamics).
5. This phenomenon appears systematically in all models of reaction-diffusion type.

All these results do not seem to be respectful with the classical comparison principle for the solutions of the heat equation.

One expects to control temperature form $19C$ to $24C$ (or viceversa), keeping temperature $19 \leq \tau \leq 24$.

This intuition is false. But we can guarantee that, if $\tau >> 1$ then:

$$18.9C \leq \tau \leq 24.1C$$

Long time horizon control for parabolic equations

Theorem

If the target ρ^1 is a positive steady state, there exists T a **large enough time** *and a* **non-negative boundary control** *$u \geq 0$ s. t. $\rho(T, \cdot) = \rho^1$. Further, if $\rho^0 \geq 0$, then, by the comparison principle, $\rho \geq 0$.*

Theorem

The minimal time is positive, i. e. there is a **positive waiting time** *to reach the target under the non-negativity constraint:*

$$T_{\text{min}} > 0.$$

And, there is a non-negative measure that assures, as control, that the target is achieved in the minimal time T_{min}.

[2] [3]

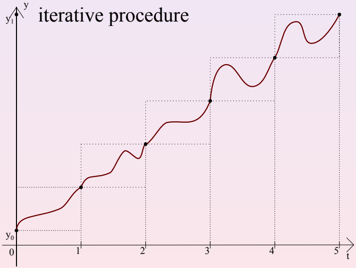
[2] J. Lohéac, E. Trélat and E. Z., M3AS, 27 (2017), no. 9, 1587-1644.

[3] D. Pighin, E. Z., MCRF & arXiv:1711.07678

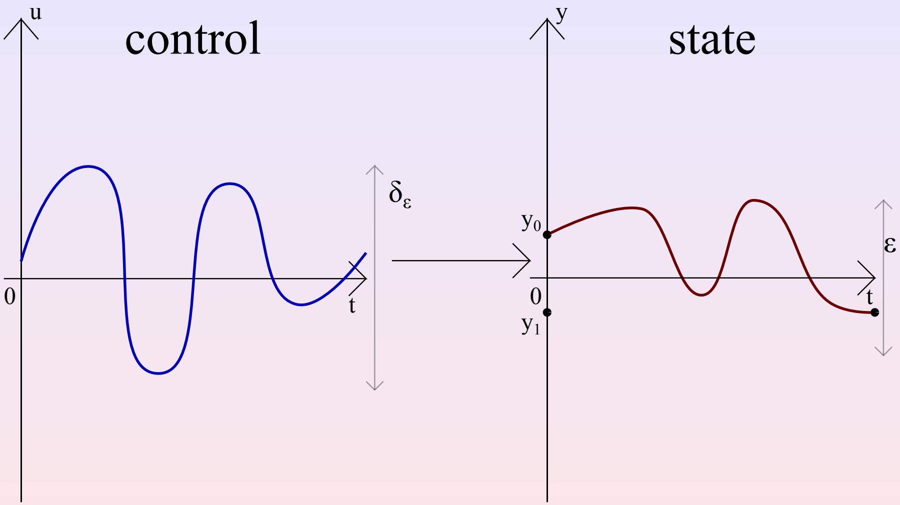
The key idea

If ρ^1 is the target steady state, the control can be kept $u \geq \rho^1 - \delta$ if the time interval is long enough $\to \infty$ ad $\delta \to 0$. Thus $u \geq 0$ if $\rho^1 > 0$.

4



4 J.-M. Coron, E. Trélat, Global Steady-State Controllability of One-Dimensional Semilinear Heat Equations, SICON 43(2), 549–569.


u
control
δ_ε
0
t
y
state
y_0
0
y_1
ε
t

Back to population dynamics

The same results hold for the age-dependent diffusive population dynamics model.

The key is to obtain L^∞-**controls of small amplitude**, which, by duality, requires to prove an observability inequality with measurements done in L^1. In other words, one has to improve the inequality

$$\int_0^{a_\dagger}\int_\Omega q^2(x,a,T)\,dx\,da \leqslant C_T \int_0^T\int_{a_1}^{a_2}\int_\omega q^2(x,a,t)\,dx\,da\,dt,$$

to

$$\int_0^{a_\dagger}\int_\Omega q^2(x,a,T)\,dx\,da \leqslant C_T \left|\int_0^T\int_{a_1}^{a_2}\int_\omega |q(x,a,t)|\,dx\,da\,dt\right|^2.$$

And this can be done since both the estimates based on transport along characteristics and in the diffusion process hold in L^1.

The impact of the impulses on the controlled trajectories

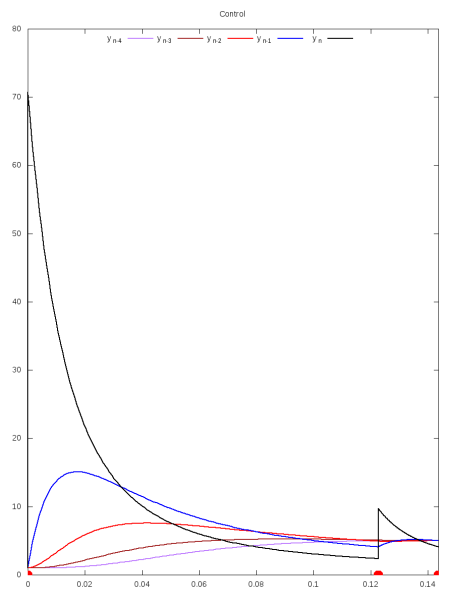


Table of Contents

Conclusions

1. A number of basic ingredients have been understood/developed.
2. Similar results for boundary control in the x variable.
3. Similar results for abstract models: replace $-\Delta$ by A (possibly in time greater than $a_1 + (a_\dagger - a_2)$).
4. Plenty of interesting open analytical problems (sparsity structure of controls in minimal time).
5. Lots to be done to address more complex and complete models in population dynamics:
   - Nonlinear diffusion
   - Systems
   - Networks
6. Delivering results which might be quantitatively accurate and useful is a true challenge.

Appendix. Constrained control for the heat equation in long time

We have shown that controllability with constraints can be achieved in long time. In fact, it is impossible to do it in short time!

Theorem

Whatever the initial datum ρ^0 and the steady state target ρ^1 associated to $u^1 \geq \nu$ is, the minimal control time under the positivity constraint is positive:

$$T_{\min} > 0,$$

except in the trivial case where $\rho^0 \equiv \rho^1$.

Proof of the waiting time

To fix ideas and without loss of generality we assume that $\rho^0 \equiv 0$.
The target $\rho^1 > 0$.
Assume that the control $u \geq 0$, and show that, then, T cannot be too short.
By duality, if $y(T) = y^1$,

$$\langle \rho^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

where

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0, T) \times \Omega \\ \varphi = 0 & \text{on } (0, T) \times \partial\Omega \\ \varphi(T, x) = \varphi^T(x). & \text{in } \Omega \end{cases}$$

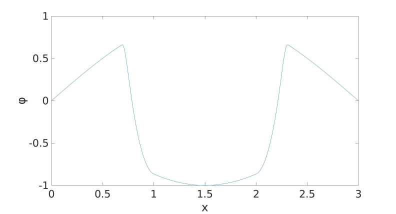
Thus, to conclude, in view of the identity

$$\langle \rho^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

it suffices to find $T_0 > 0$ and a final datum $\varphi^T \in H_0^1(\Omega)$ such that, for any $T \in (0, T_0)$, the solution of the adjoint system with final datum φ^T satisfies:

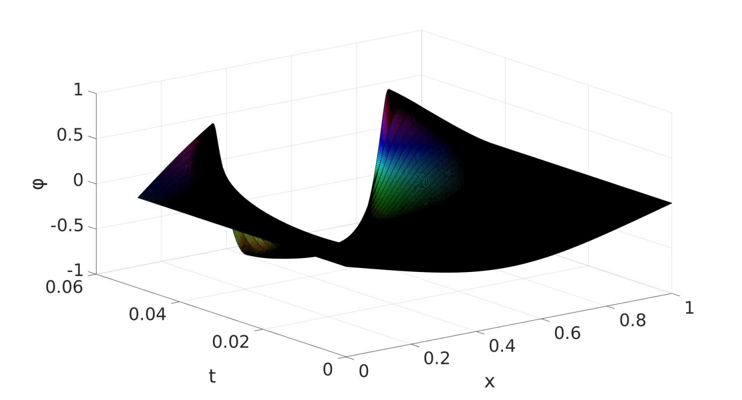
$$\begin{cases} \left(\frac{\partial \varphi}{\partial n}\right)_+ = 0 & \text{on } (0, T_0) \times \partial\Omega \\ \langle \rho^1, \varphi^T \rangle < 0, & \forall T \in [0, T_0). \end{cases}$$

This is assured with an initial datum of the form



Explicit estimates on the waiting time

The proof above not only yields the fact that $T_{\min} > 0$, but actually gives lower bounds on this waiting time, by a careful analysis of the behaviour of the adjoint solutions.



Controllability in minimal time

Theorem

The system is controllable in minimal time $T_{\min}$ with a non-negative measure u as control

The proof is again a consequence of the identity:

$$\langle \rho^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

now applied to the solution of the adjoint heat equation φ with Φ_1, the first eigenfunction of the Laplacian, as datum at time T:

$$\varphi = \exp(-\lambda_1(T - t))\Phi_1(x).$$

This allows getting an L^1 bound on the control and then get a limite measure control as $T \to T_{\text{min}}$.