Flow control in the presence of shocks

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Outline

Introduction

Flow control, as many other fields of applied mathematics involves:

- Partial Differential Equations: Models describing motion in the various fields of Mechanics: Elasticity, Fluids,...
- Numerical Analysis: Allowing to discretize these models so that solutions may be approximated algorithmically.
- Optimal Design: Design of shapes to enhance the desired properties (bridges, dams, aeroplanes,..)
- Control: Automatic and active control of processes to guarantee their best possible behavior and dynamics.

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These topics meet together in many relevant applications.

- Noise reduction in cavities and vehicles.
- Laser control in quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: the Thames barrier.
- Optimal shape design in aeronautics.
- Human cardiovascular system.
- Oil prospection and recovery.
- Irrigation systems.

Complexity arises in many ways:

- Geometry;
- Oscillations.

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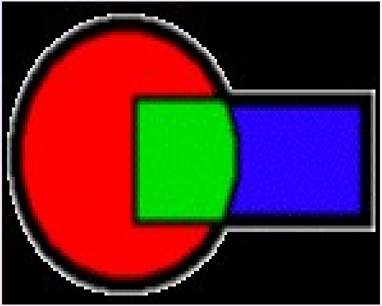
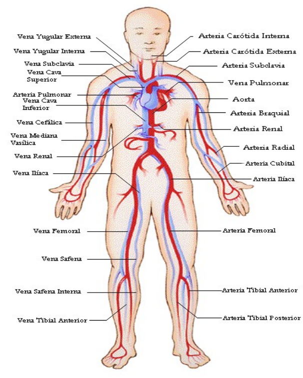
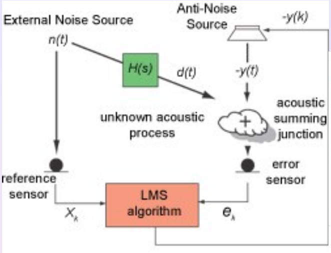




The logo of the web page "Domain decomposition", one of the most widely used computational techniques for solving PDE in domains ("divide y vencerás"), and a drawing of the human cardiovascular system illustrating the graph along which blood circulates.


External Noise Source
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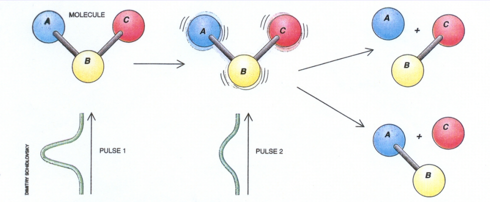
MOLECULE
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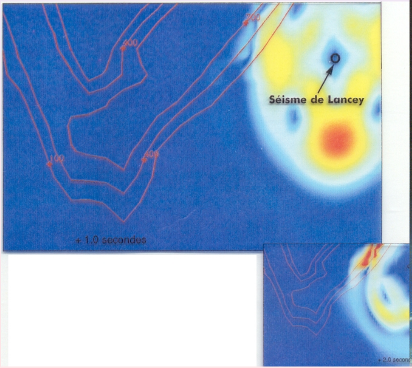
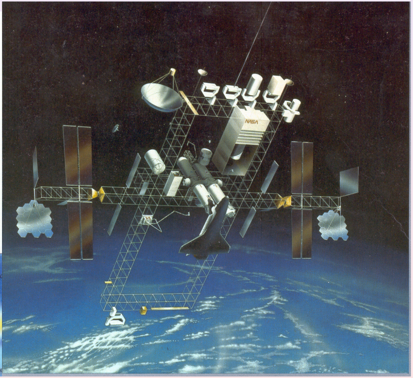
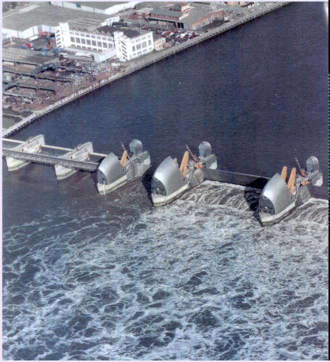
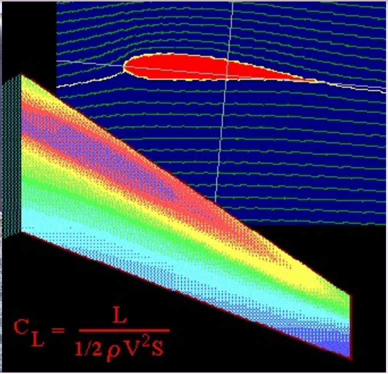
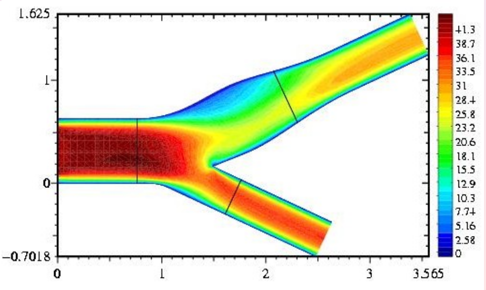

Séisme de Lancey
+ 1.0 secondes






$C_L = \frac{L}{1/2 \rho V^2 S}$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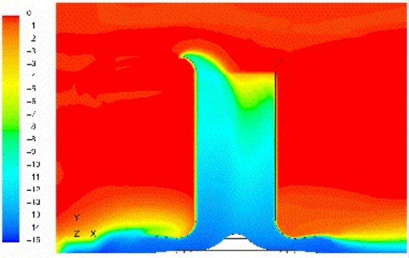
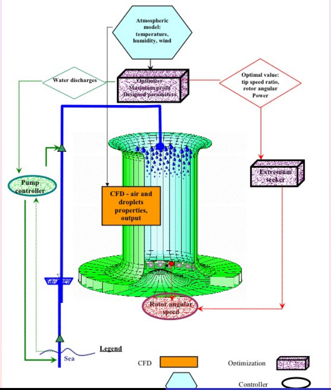
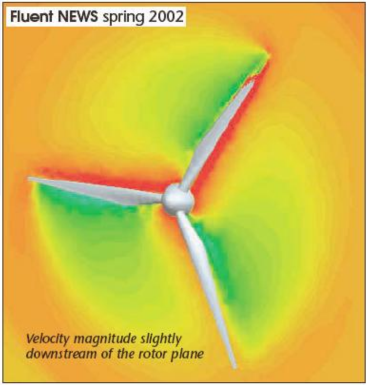
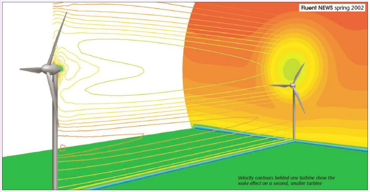
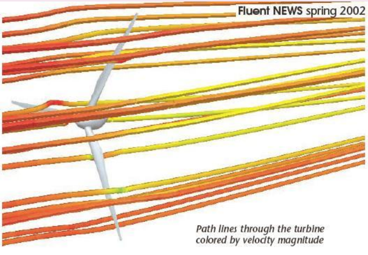
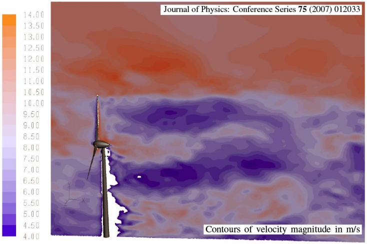



Atmospheric model: temperature, humidity, wind
Water discharges
Optimizer- Maximum profit Designed parameters
Optimal value: tip speed ratio, rotor angular Power
Extremum seeker
Pump controller
CFD - air and droplets properties, output
Rotor angular speed
Sea
Legend
CFD
Optimization
Controller


Fluent NEWS spring 2002
Velocity magnitude slightly
downstream of the rotor plane


Fluent NEWS spring 2002
Velocity contours behind one turbine show the
wake effect on a second, smaller turbine


Fluent NEWS spring 2002
Path lines through the turbine
colored by velocity magnitude


Journal of Physics: Conference Series 75 (2007) 012033
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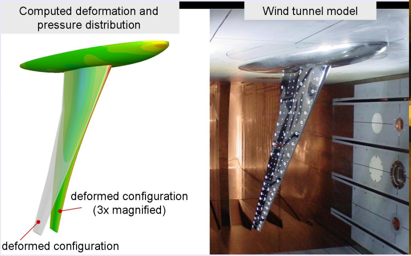
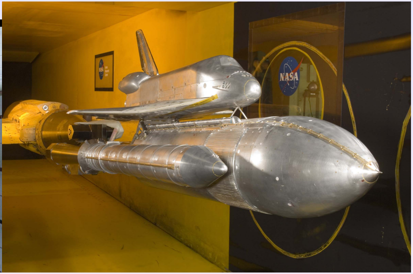
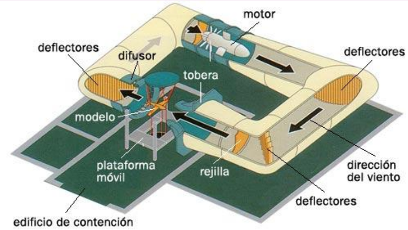
Contours of velocity magnitude in m/s


Computed deformation and pressure distribution
Wind tunnel model
deformed configuration (3x magnified)
deformed configuration




motor
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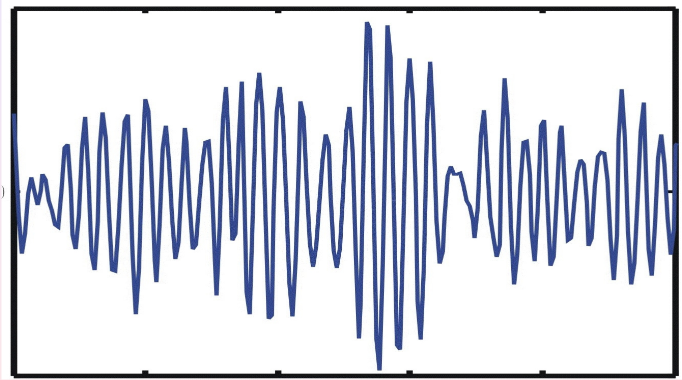
tobera
modelo
plataforma móvil
rejilla
dirección del viento
deflectores
edificio de contención

Computer simulation $\rightarrow$ far beyond the fields in which its use is justified (consistency + stability $\equiv$ convergence).

The risk: To end up getting numerical data whose validity....



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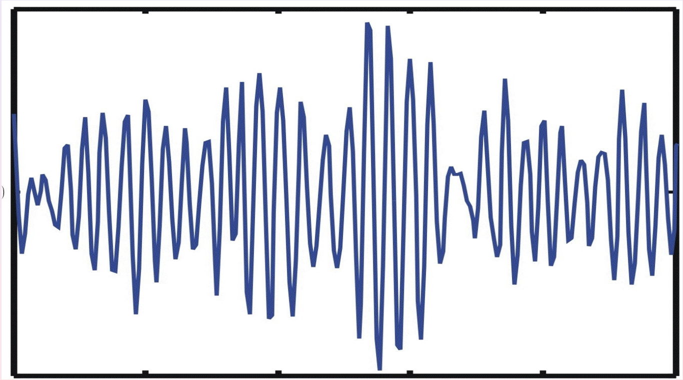
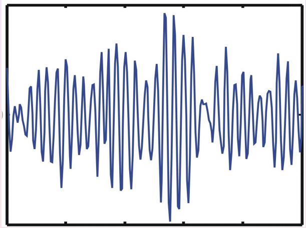


Is this difficulty solvable in practice?

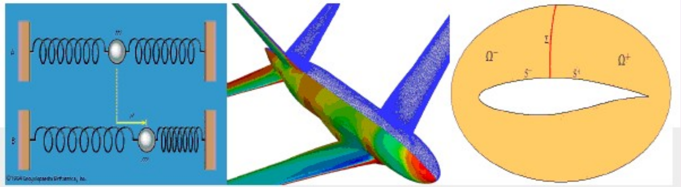
- Solvable for problems with well known data.
- Much harder for inverse, design and control problems,,,,

In those cases the obtained final numerical results and simulations may simply mean nothing.



Optimal shape design in aeronautics. Two aspects:

- Shocks.
- Oscillations.



Optimal shape ∼ Active control. The shape of the cavity or airfoil controls the surrounding flow of air.

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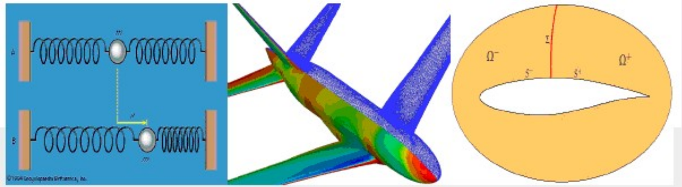

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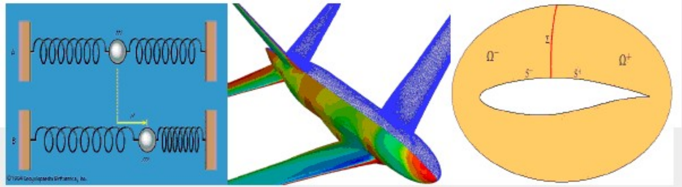

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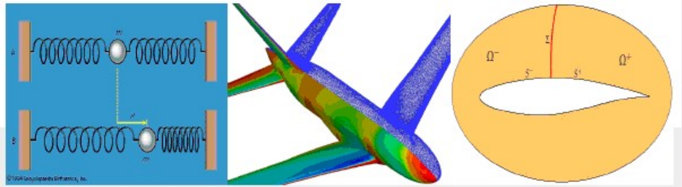

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- Aeronautics: to simulate and optimize complex processes is indispensable.
- Long tradition: J. L. Lions, A. Jameson,...
- However, this needs an immense computational effort.

For practical optimization problems, in which at least 100 design variables are to be considered, current methodological approaches applied in industry will need more than a year to obtain an optimized aircraft.

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Minimize

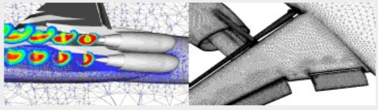
$$J(\Omega^*) = \min_{\Omega \in \mathcal{C}_{ad}} J(\Omega)$$

$\mathcal{C}_{ad}$ = class of admissible domains.

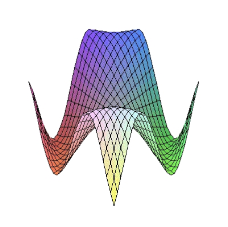
J = cost functional (drag reduction, lift maximization, exploitation cost, overall cost over the life cycle of the aircraft, benefit maximization, etc).

J depends on Ω through $u(\Omega)$, solution of the PDE (elasticity, Fluid Mechanics,...).

The domains under consideration are often complex. Geometric and parametrization issues play a key role.



The dependence of the functional on the domain, through the solution of the PDE is complex as well. J it is far from being a nice convex function.



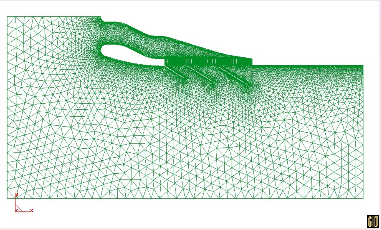
Analytical difficulties:

- Lack of good existence, uniqueness, and continuous dependence theory for the PDE.
- Lack of convexity of the functional.
- Lack of compactness within the class of relevant domains...

In practice

Descent algorithm (gradient based method) on a discrete version of the problem:

- The domains Ω have been discretized (finite element mesh)
- The PDE has been replaced by a numerical scheme,
- The functional J has been replaced by a discrete version.



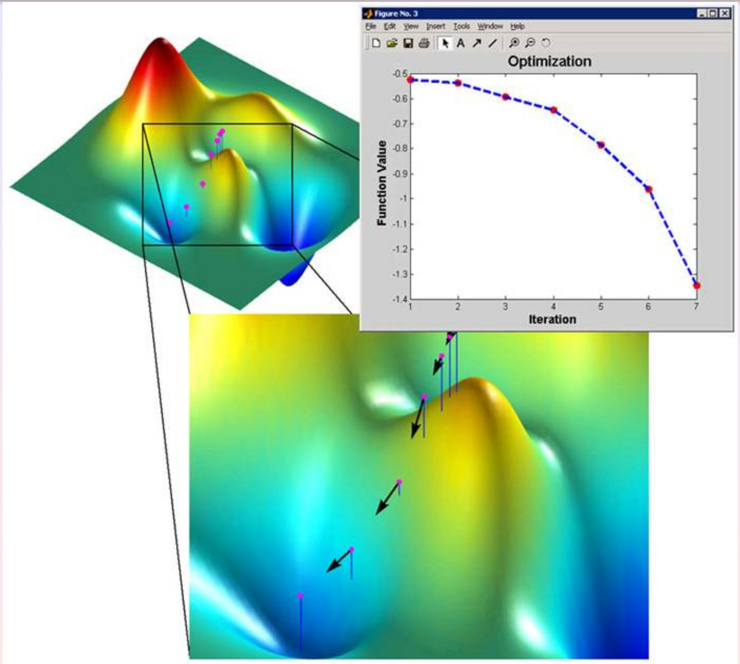


Figure No. 3
File Edit View Insert Tools Window Help
Optimization
Function Value
Iteration
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1.1
-1.2
-1.3
-1.4
1
2
3
4
5
6
7

Classical steepest descent:

$J : H \to \mathbf{R}$. Two main assumptions:

$$< \nabla J(u) - \nabla J(v), u - v > \geq \alpha |u - v|^2, \quad |\nabla J(u) - \nabla J(v)|^2 \leq M |u - v|^2.$$

Then, for

$$u_{k+1} = u_k - \rho \nabla J(u_k),$$

we have

$$|u_k - u^*| \leq (1 - 2\rho\alpha + \rho^2 M)^{k/2} |u_1 - u^*|.$$

Convergence is guaranteed for $0 < \rho < 1$ small enough.

Compare with the continuous marching gradient system

$$u'(\tau) = -\nabla J(u(\tau)).$$

We end up with:

- A discrete optimization problem of huge dimensions,
- No idea of whether discrete optima, if they exist, will converge or not to the optimal continuous one.
  - Analytical difficulties.
  - Divergence of algorithms.

The worst scenario: When using results provided by divergent algorithms, for which divergence is hard to detect.

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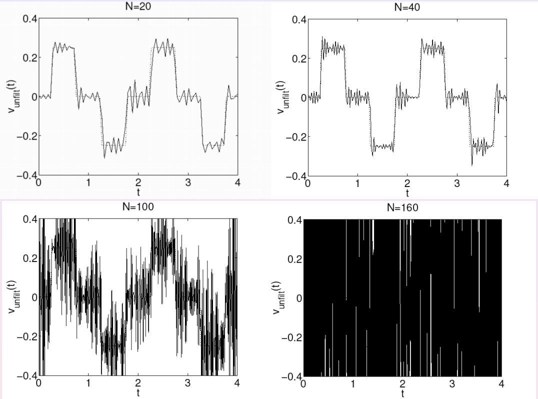
An example: boundary control of vibrations.



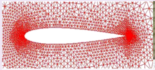
Can we guarantee this kind of pathologies do not arise in realistic problems of optimal shape design in aeronautics?

How to detect them? How to avoid them?

Two approaches:

Discrete: Discretization + gradient

- Advantages: Discrete clouds of values. No shocks. Automatic differentiation, ...
- Drawbacks:
  - "Invisible" geometry.



  - Scheme dependent.

Continuous: Continuous gradient + discretization.

- Advantages: Simpler computations. Solver independent. Shock detection.
- Drawbacks:
  - Yields approximate gradients.
  - Subtle if shocks.

The relevant models in aeronautics (Fluid Mechanics):

- Navier-Stokes equations;
- Euler equations;
- Turbulent models: Reynolds-Averaged Navier-Stokes (RANS), Spalart-Allmaras Turbulence Model, $k - \varepsilon$ model;

  ....
- Burgers equation (as a $1 - d$ theoretical laboratory).

Euler equations

$$\begin{cases} \partial_t U + \vec{\nabla} \cdot \vec{F} = 0, & \text{in } \Omega, \\ \vec{v} \cdot \vec{n}_S = 0, & \text{on } S, \end{cases}$$

with suitable boundary conditions at infinity,
$U = (\rho, \rho v_x, \rho v_y, \rho E)$= conservative variables, $\vec{F} = (F_x, F_y)$=flux

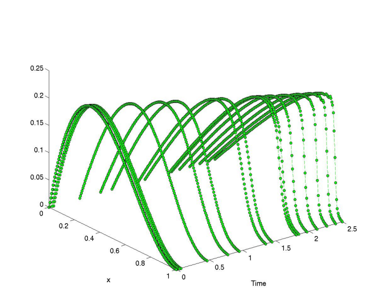
$$F_x = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + P \\ \rho v_x v_y \\ \rho v_x H \end{pmatrix}, \; F_y = \begin{pmatrix} \rho v_y \\ \rho v_x v_y \\ \rho v_y^2 + P \\ \rho v_y H \end{pmatrix}, \qquad (1)$$

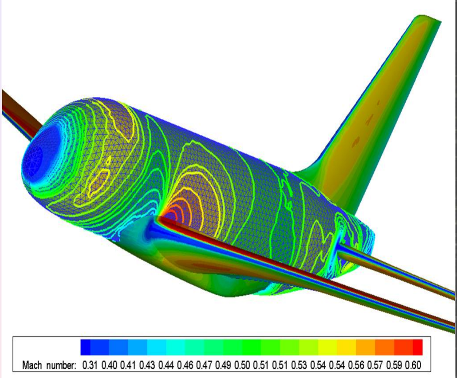
$\rho =$ density , $\vec{v} = (v_x, v_y) =$ velocity, $E =$ total energy, $P =$ pressure, $H =$ enthalpy, where

$$P = (\gamma - 1)\rho \left(E - \frac{1}{2}(u^2 + v^2) \right), \quad H = E + \frac{P}{\rho}.$$

Solutions may develop shocks or quasi-shock configurations.

- For shock solutions, classical calculus fails;
- For quasi-shock solutions the sensitivity is so large that classical sensitivity clalculus is meaningless.

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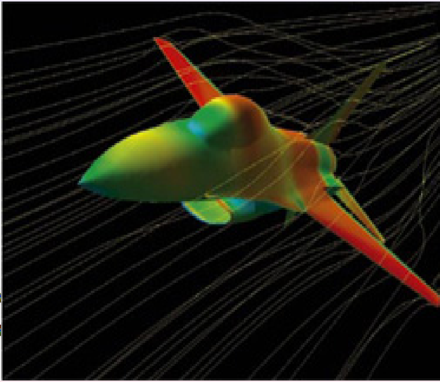


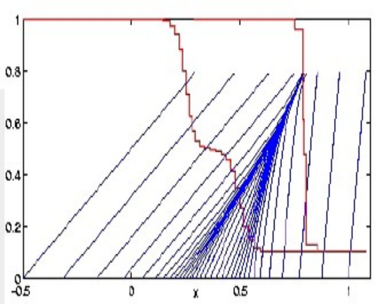
Burgers equation

- Viscous version:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0.$$

- Inviscid one:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$



In the inviscid case, the simple and "natural" rule

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \rightarrow \frac{\partial \delta u}{\partial t} + \delta u\frac{\partial u}{\partial x} + u\frac{\partial \delta u}{\partial x} = 0$$

breaks down in the presence of shocks

$\delta u =$ discontinuous, $\frac{\partial u}{\partial x} =$ Dirac delta $\Rightarrow \delta u \frac{\partial u}{\partial x}$????

The difficulty may be overcame with a suitable notion of measure valued weak solution using Volpert's definition of conservative products and duality theory (Bouchut-James, Godlewski-Raviart,...)

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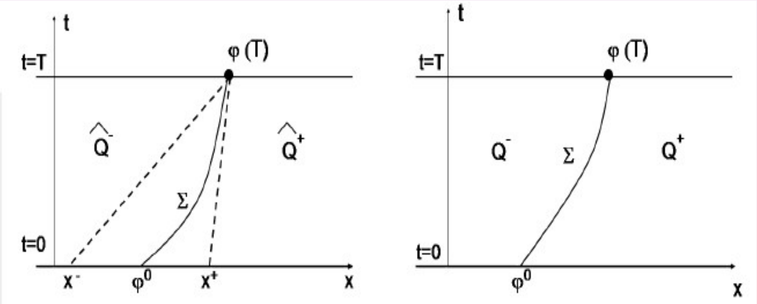
A new viewpoint: Solution = Solution + shock location. Then the pair (u, φ) solves:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, & \text{in } Q^- \cup Q^+, \\ \varphi'(t)[u]_{\varphi(t)} = \left[u^2/2\right]_{\varphi(t)}, & t \in (0, T), \\ \varphi(0) = \varphi^0, & \\ u(x,0) = u^0(x), & \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}. \end{cases}$$



The corresponding linearized system is:

$$\begin{cases} \partial_t \delta u + \partial_x (u \delta u) = 0, \quad \text{in } Q^- \cup Q^+, \\ \\ \delta\varphi'(t)[u]_{\varphi(t)} + \delta\varphi(t)\left(\varphi'(t)[u_x]_{\varphi(t)} - [u_x u]_{\varphi(t)}\right) \\ \quad +\varphi'(t)[\delta u]_{\varphi(t)} - [u\delta u]_{\varphi(t)} = 0, \quad \text{in } (0,T), \\ \\ \delta u(x,0) = \delta u^0, \quad \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}, \\ \delta\varphi(0) = \delta\varphi^0, \end{cases}$$

Majda (1983), Bressan-Marson (1995), Godlewski-Raviart (1999), Bouchut-James (1998), Giles-Pierce (2001), Bardos-Pironneau (2002), Ulbrich (2003), ...
None seems to provide a clear-cut recipe about how to proceed within an optimization loop.

A new method: Splitting + alternating descent algorithm.
C. Castro, F. Palacios, E. Z., M3AS, to appear.
Ingredients:

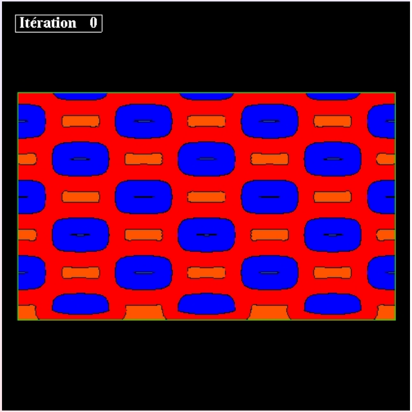
- The shock location is part of the state.

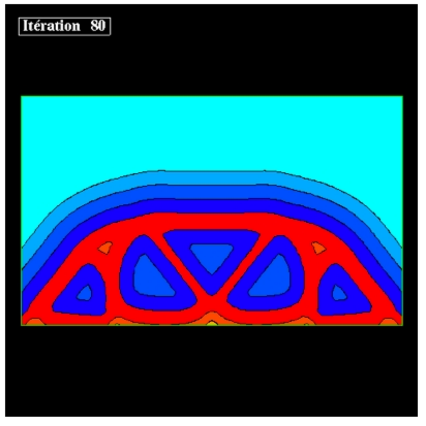
  State = Solution as a function + Geometric location of shocks.

- Alternate within the descent algorithm:
  - Shock location and smooth pieces of solutions should be treated differently;
  - When dealing with smooth pieces most methods provide similar results;
  - Shocks should be handeled by geometric tools, not only those based on the analytical solving of equations.

Lots to be done: Pattern detection, image processing, computational geometry,... to locate, deform shock locations,....

Compare with the use of shape and topological derivatives in elasticity:





Consider

$$J(u^0) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

u^d = step function.

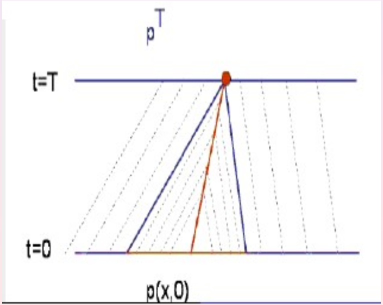
Gateaux derivative:

$$\delta J = \int_{\{x<\varphi^0\}\cup\{x>\varphi^0\}} p(x,0)\delta u^0(x)\, dx + q(0)[u]_{\varphi^0}\delta\varphi^0,$$

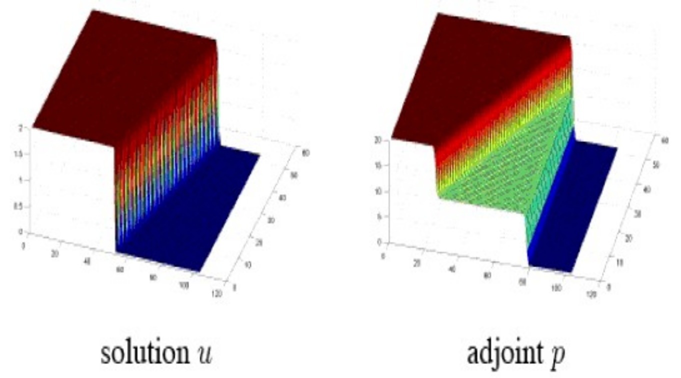
(p, q) = adjoint state

$$\begin{cases} -\partial_t p - u\partial_x p = 0, \quad \text{in } Q^- \cup Q^+, \\ [p]_\Sigma = 0, \\ q(t) = p(\varphi(t), t), \ \text{in } t \in (0,T) \\ q'(t) = 0, \ \text{in } t \in (0,T) \\ p(x,T) = u(x,T) - u^d, \quad \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\} \\ q(T) = \frac{\frac{1}{2}\left[(u(x,T)-u^d)^2\right]_{\varphi(T)}}{[u]_{\varphi(T)}}. \end{cases}$$

- The gradient is twofold= variation of the profile + shock location.
- The adjoint system is the superposition of two systems = Linearized adjoint transport equation on both sides of the shock + Dirichlet boundary condition along the shock that propagates along characteristics and fills all the region not covered by the adjoint equations.



State u and adjoint state p when u develops a shock:



The discrete aproach

Recall the continuous functional

$$J(u^0) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

The discrete version:

$$J^\Delta(u_\Delta^0) = \frac{\Delta x}{2}\sum_{j=-\infty}^{\infty}(u_j^{N+1} - u_j^d)^2,$$

where $u_\Delta = \{u_j^k\}$ solves the 3-point conservative numerical approximation scheme:

$$u_j^{n+1} = u_j^n - \lambda\left(g_{j+1/2}^n - g_{j-1/2}^n\right) = 0, \quad \lambda = \frac{\Delta t}{\Delta x},$$

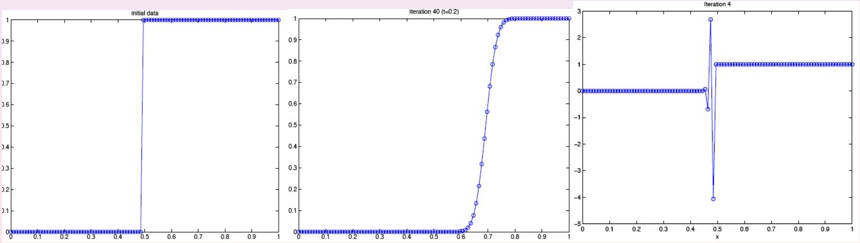
where, g is the numerical flux

$$g_{j+1/2}^n = g(u_j^n, u_{j+1}^n), \, g(u,u) = u^2/2.$$

Examples of numerical fluxes

$$
\begin{aligned}
g^{LF}(u,v) &= \frac{u^2+v^2}{4} - \frac{v-u}{2\lambda}, \\
g^{EO}(u,v) &= \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4}, \\
g^{G}(u,v) &= \begin{cases} \min_{w\in[u,v]} w^2/2, & \text{if } u \le v, \\ \max_{w\in[u,v]} w^2/2, & \text{if } u \ge v, \end{cases}
\end{aligned}
$$

Instabilities arise immediately, even for the lineat transport equation $u_t + u_x = 0$ whose solution is $u = f(x - t)$, if the numerical scheme does not obey the “upwind” law. Note that at the discrete level there are several ways to approximate u_x: centered $(\frac{u_{j+1} - u_{j-1}}{2\Delta x})$; forward $(\frac{u_{j+1} - u_j}{\Delta x})$; and backward $(\frac{u_j - u_{j-1}}{\Delta x})$. This matters!



The Γ-convergence of discrete minimizers towards continuous ones is guaranteed for the schemes satisfying the so called one-sided Lipschitz condition (OSLC):

$$\frac{u_{j+1}^n - u_j^n}{\Delta x} \leq \frac{1}{n\Delta t},$$

which is the discrete version of the Oleinick condition for the solutions of the continuous Burgers equations

$$u_x \leq \frac{1}{t},$$

which excludes non-admissible shocks and provides the needed compactness of families of bounded solutions.

As proved by Brenier-Osher, [1] Godunov's, Lax-Friedfrichs and Engquits-Osher schemes fulfil the OSLC condition.

[1] Brenier, Y. and Osher, S. The Discrete One-Sided Lipschitz Condition for Convex Scalar Conservation Laws, SIAM Journal on Numerical Analysis, **25** (1) (1988), 8-23.

A new method: splitting+alternating descent

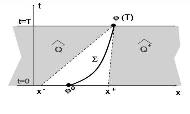
- Generalized tangent vectors $(\delta u^0, \delta\varphi^0) \in T_{u^0}$ s. t.

$$\delta\varphi^0 = \left(\int_{x^-}^{\varphi^0} \delta u^0 + \int_{\varphi^0}^{x^+} \delta u^0 \right) \Big/ [u]_{\varphi^0}.$$

do not move the shock $\delta\varphi(T) = 0$ and

$$\delta J = \int_{\{x<x^-\}\cup\{x>x^+\}} p(x,0)\delta u^0(x)\, dx,$$

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } \hat{Q}^- \cup \hat{Q}^+, \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\}. \end{cases}$$



For those descent directions the adjoint state can be computed by "any numerical scheme"!

- Analogously, if $\delta u^0 = 0$, the profile of the solution does not change, $\delta u(x, T) = 0$ and

$$\delta J = -\left[\frac{(u(x,T) - u^d(x))^2}{2}\right]_{\varphi(T)} \frac{[u^0]_{\varphi^0}}{[u(\cdot, T)]_{\varphi(T)}} \delta\varphi^0.$$

This formula indicates whether the descent shock variation is left or right!

WE PROPOSE AN ALTERNATING STRATEGY FOR DESCENT

In each iteration of the descent algorithm do two steps:

- Step 1: Use variations that only care about the shock location
- Step 2: Use variations that do not move the shock and only affect the shape away from it.

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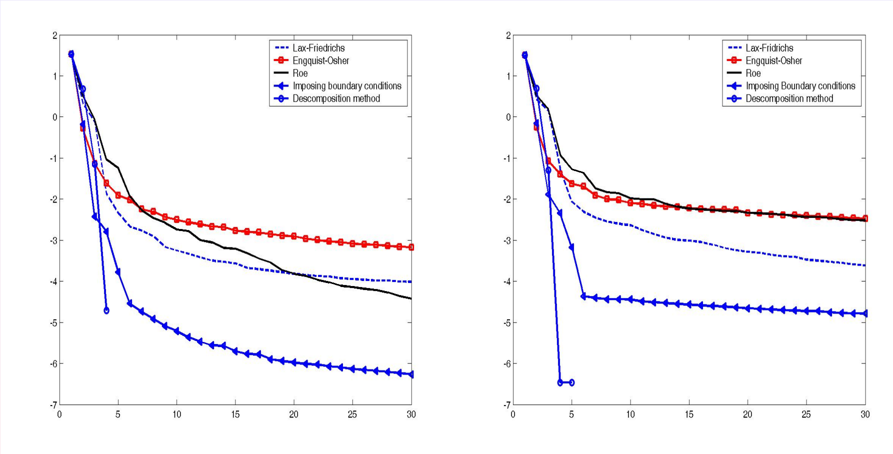
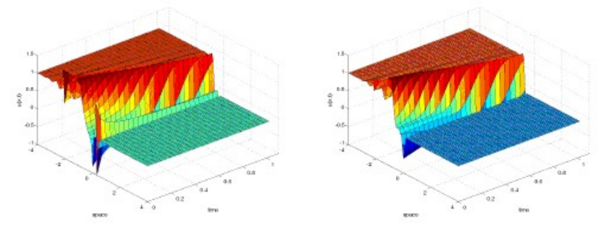
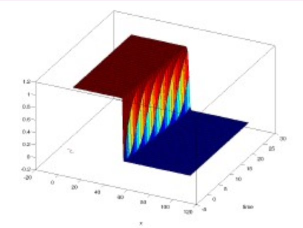


Splitting+Alternating wins!

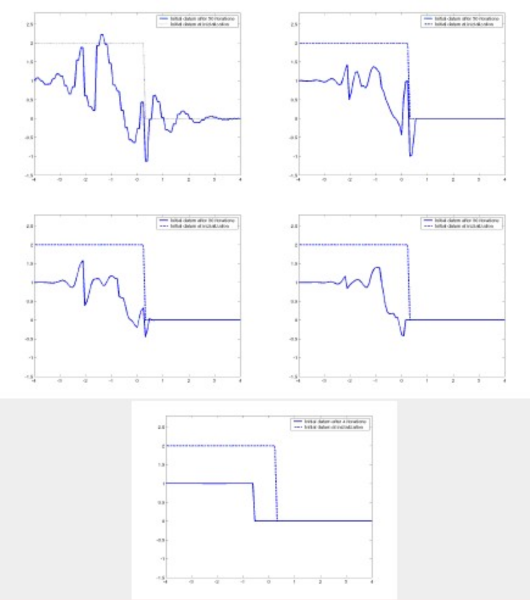


Results obtained applying Engquist-Osher's scheme and the one based on the complete adjoint system



Splitting+Alternating method.

After 30 iterations:

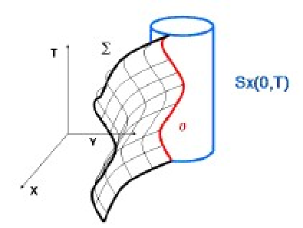


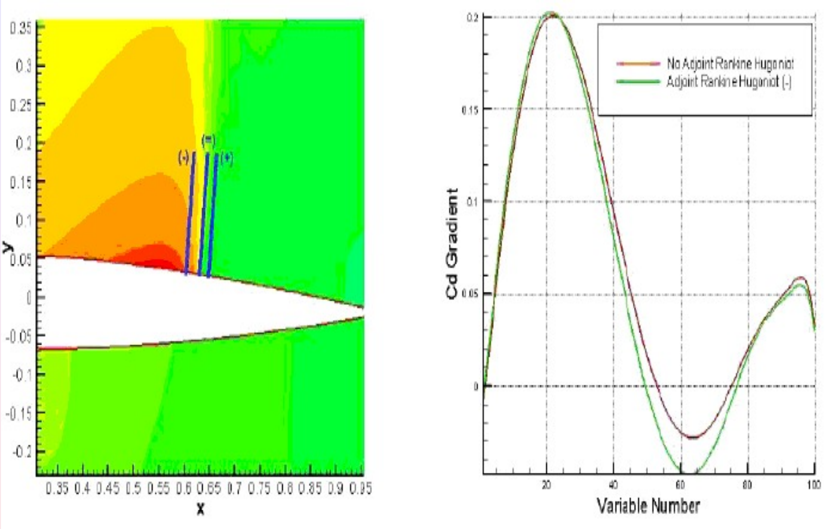
Splitting+alternating is more efficient:

- It is faster.
- It does not increase the complexity.
- Rather independent of the numerical scheme.

Extending these ideas and methods to more realistic multi-dimensional problems is a work in progress and much remains to be done.

Numerical schemes for PDE + shock detection + shape, shock deformation + mesh adaptation,...





Influence of shock wave location (Drag Minimization).

- Much remains to be done in the interfaces between PDE, numerical analysis and optimal design:
  - Well-posedness of relevant models;
  - New approximation schemes for linearized and adjoint equations;
  - Rigorous proof of convergence of new descent algorithms (shock handeling, regularization,...)
- An important effort has to be done to bring all this mathematical understanding and theory to real applications: Make all this to become algorithmic and insert it into the relevant software to be used in (in particular) aeronautical engineering.

Thanks to: E. Andrés, T. Baeza, C. Castro, M. Cea, L. Ignat, C. Lozano, A. Marica, F. Palacios, ...

Let $u = u(x, t)$ be a solution of

$$u_t - \nu u_{xx} + (u^2)_x = 0.$$

such that $| u(x, t) | + | u_x(x, t) | \to 0$ as $| x | \to \infty$.
Then

$$v = v(x, t) = \int_{-\infty}^{x} u(s, t) ds \quad (2)$$

solves

$$v_t - \nu v_{xx} + | v_x |^2 = 0. \quad (3)$$

Define then

$$w = v(x, t/\nu)$$

that satisfies

$$w_t - w_{xx} + \frac{1}{\nu} | w_x |^2 = 0. \quad (4)$$

On the other hand,

$$z = w/\nu \tag{5}$$

satisfies

$$z_t - z_{xx} + \mid z_x \mid^2 = 0. \tag{6}$$

Introduce, at last,

$$\eta(x,\, t) = e^{-z} \tag{7}$$

that solves the heat equation

$$\eta_t - \eta_{xx} = 0. \tag{8}$$

Undoing the change of variables

$$\begin{aligned} & u = v_x \\ & v(\cdot,\, t/\nu) = w(\cdot,\, t) = \nu z(\cdot,\, t) = -\nu \log(\eta). \end{aligned}$$

Then

$$u(x,\, t) = -\nu \frac{\eta_x(x,\, \nu t)}{\eta(x,\, \nu t)}. \tag{9}$$

The solution η of this heat equation can be obtained by convolution with the heat kernel:

$$G(x,\ t) = (4\pi t)^{-1/2} \exp\left(- \mid x \mid^2 \Big/ 4t\right), \tag{10}$$

so that

$$\eta(x,\ t) = \Big[G(\cdot,\ t) * \eta_0(\cdot)\Big](x), \tag{11}$$

where η_0 is the initial datum of η.
On the other hand,

$$G_x(x,\ t) = -\frac{x}{4\sqrt{\pi}t^{3/2}} \exp\left(- \mid x \mid^2 \Big/ 4t\right). \tag{12}$$

In this way we get

$$u_\nu(x,\ t) = \frac{\int_{\mathbb{R}} (x-y) e^{-H(x,y,t)/\nu} dy}{2t \int_{\mathbb{R}} e^{-H(x,y,t)/\nu} dy} \tag{13}$$

where

$$H(x,\ y,\ t) = \frac{\mid x-y \mid^2}{4t} + \int_{-\infty}^{y} u_0(\sigma) d\sigma. \tag{14}$$

The contribution of the integral as $\nu \to 0^+$[2]

$$\int_{\mathbb{R}} f(y) e^{-H/\nu} dy \tag{15}$$

around the minimum $y = \xi$ is

$$f(\xi) \sqrt{\frac{2\pi\nu}{H''(\xi)}} e^{-H(\xi)/\nu}. \tag{16}$$

In our case

$$H''(\xi) = \frac{1}{2t}. \tag{17}$$

We get

$$\int_{\mathbb{R}} (x - y) e^{-H/\nu} dy \sim (x - \xi) \sqrt{\frac{\pi\nu}{t}} e^{-[t u_0^2(\xi) + \int_{-\infty}^{\xi} u_0(\sigma) d\sigma]\nu}, \tag{18}$$

$$\int_{\mathbb{R}} e^{-H/\nu} dy \sim \sqrt{\frac{\pi\nu}{t}} e^{-[t u_0^2(\xi) + \int_{-\infty}^{\xi} u_0(\sigma) d\sigma]}. \tag{19}$$

[2] Carl M. Bender and Steven A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory , Springer, 1999

Then

$$u_\nu(x,\ t) \sim \frac{(x-\xi)}{2t} \tag{20}$$

where ξ is characterized by the equation

$$\xi = x - 2tu_0(\xi), \tag{21}$$

corresponding to the minima of H.
Thus

$$u_\nu(x,\ t) \sim u_0(\xi). \tag{22}$$

This is, precisely, the solution obtained by the method of characteristics:

$$u_\nu(x,\ t) \sim u_0(\xi). \tag{23}$$

This is valid when H has only one minimum.

When u_0 is increasing and smooth there is only one solution and we recover the same solution as the one obtained by the method of characteristics.

When H has several minima $\xi_1, \ldots, \xi_N$, each of them provides a contribution of the same form.

When there are two absolute minima ξ_1, ξ_2, the asymptotic form of u_ν would be:

$$u_\nu(x, t) \sim u_0(\xi_1) + u_0(\xi_2). \tag{24}$$

What does it mean?

We now consider the Riemann problem

$$u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases} \quad (25)$$

We get:

- When $x < 0$, this gives $\xi = x$ and then the limit is $u = u_0(\xi) = 0$.
- When $x > 2t$ we get $\xi = x - 2t$ and then the solution is $u \equiv 1$, which coincides with the result that the method of characteristic yields.
- In the intermediate zone we get $\xi = x/(1 + 2t)$ and

$$u(x,\ t) = \frac{x}{2t}. \quad (26)$$

The rarefaction wave $u = x/2t$ connects the value $u = 0$ to the left and $x = 1$ to the right.
This is the physical or entropy solution.

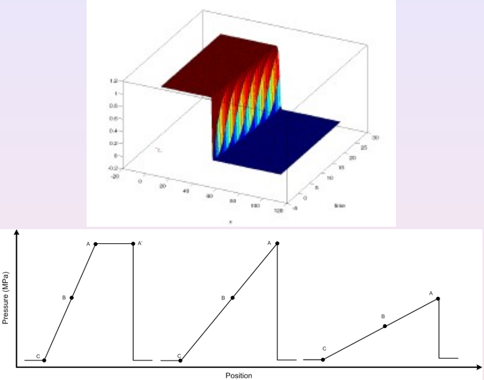
For the Riemann problem

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \tag{27}$$

we get a shock like solution.

The method of vanishing viscosity confirms this is the entropy or physical solution.

In this case, the function H has two local minima, but when determining the global one we get either the value $u \equiv 0$ or $u \equiv 1$ depending on whether we are on the left or right of the shock.



Shock versus rarefaction waves

Physical solutions of the Burgers equation satisfy

$$u_x \leq 1/2t. \tag{28}$$

Formally, $v = u_x$ satisfies

$$v_t + (2uv)_x = v_t + 2v^2 + 2uv_x = 0. \tag{29}$$

By the maximum principle we deduce that

$$v \leq w \tag{30}$$

where $w = w(t)$ is the solution of

$$w_t + 2w^2 = 0 \tag{31}$$

with initial datum $w(0) = \infty$: $w(t) = 1/2t$.
This formal argument can be fully justified for the physical solutions that are obtained as zero viscosity limits.

Note that this is compatible with the structure of shock waves

Summary on entropy solutions of the Burgers equation

- Entropy solutions are the physical ones
- Entropy solutions are characterized by the zero viscosity limit.
- Entropy solutions are characterized also by the Oleinick inequality.
- Entropy solutions are unique (celebrated result by Kruzkov).

All this can be extended to multi-dimensional scalar conservation laws:

$$u_t + \mathrm{div}(\vec{f}(u)) = 0.$$

Note however that, in real applications, we often deal with systems, where theory if much more complex and only partially complete.