Inverse time design

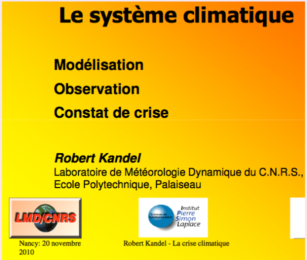
Enrique Zuazua

April 3, 2020

- Hyperbolic/parabolic PDE
- Control
- Inverse problems
- Long time
- Irreversibility (diffusion+shocks)

Climate modelling

- Climate modeling is a grand challenge computational problem, a research topic at the frontier of computational science.
- Simplified models for geophysical flows have been developed with the aim to capture the important geophysical structures, while keeping the computational cost at a minimum.
- Although successful in numerical weather prediction, these models have a prohibitively high computational cost in climate modeling.



Thames barrier

- The Thames Barrier's purpose is to prevent London from being flooded by exceptionally high tides and storm surges.
- A storm surge generated by low pressure in the Atlantic Ocean, past the north of Scotland may then be driven into the shallow waters of the North Sea. The surge tide is funnelled down the North Sea which narrows towards the English Channel and the Thames Estuary. If the storm surge coincides with a spring tide, dangerously high water levels can occur in the Thames Estuary. This situation combined with downstream flows in the Thames provides the triggers for flood defence operations.

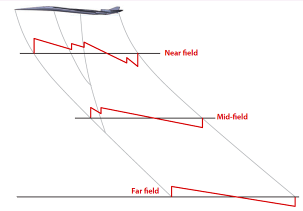
Tsunamis

- Some isolated waves (solitons) are large and travel without loss of energy.
- This is the case of tsunamis and rogue waves.

Warning: Hence, there is no use trying sending a counterwave to stop a tsunami!

Sonic boom

- Goal: the development of supersonic aircraft that are sufficiently quiet so that they can be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when it reaches the ground, (a) it can barely be perceived by the human ear, and (b) it results in disturbances to man-made structures that do not exceed the threshold of annoyance for a significant percentage of the population.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 26.

Backward resolution

It is as "simple" as solving the equation

$$u_t + A(u) = 0$$

backwards in time:

$$u(T) \to u(0)$$

The wave equation

Consider:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \end{cases}$$

Conservation of the energy:

$$E(0) = E(T),$$

or, equivalently,

$$E(T) = E(0).$$

The equation, being well-posed in the backward sense, the inverse design problem has a unique solution, living in the same space as the target is.

The heat equation

Consider:

$$\begin{cases} y_t - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \end{cases}$$

Highly dissipative effect:

$$||y(T)||^2_{L^2(\Omega)} = \sum_k e^{-2\lambda_k T} |\hat{y}^0_k|^2$$

This can also be re-written in terms of the energy of the initial data (At $t = 0$) recovered out of the final value at $t = T$:

$$\sum_k e^{-2\lambda_k T} |\hat{y}^0_k|^2 = ||y(T)||^2_{L^2(\Omega)}.$$

The same occurs for the solution of the Cauchy problem in the whole space:

$$y(x, t) = [G(\cdot, t) * y(\cdot)](x);\, G(x, t) = (4\pi t)^{-N/2} \exp(-|x|^2/4t).$$

A strongly irreversible process too.

Ill-posed in the sense of Hadamard (1865-1963)

The mathematical term **well-posed problem** stems from a definition given by Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that

1. A solution exists
2. The solution is unique
3. The solution's behavior changes continuously with the initial conditions.

$$\sum_k e^{-\lambda_k T} |\hat{y}_k^0|^2 = ||y(T)||^2_{L^2(\Omega)}.$$

Backward uniqueness

The energy identity yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \int_\Omega \mid \nabla y \mid^2 dx = 0, \tag{1}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \mathbf{\Lambda(t)} \parallel y(t) \parallel^2_{L^2(\Omega)} = 0, \tag{2}$$

$$\mathbf{\Lambda(t)} = \parallel \nabla y(t) \parallel^2_{L^2(\Omega)} \parallel y(t) \parallel^2_{L^2(\Omega)} . \tag{3}$$

The key observation is that

$$\mathbf{\Lambda(t)} \leq \mathbf{\Lambda(0)} = \mathbf{\Lambda_0}, \; \forall t \geq 0.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \mathbf{\Lambda_0} \parallel y(t) \parallel^2_{L^2(\Omega)} \geq 0. \tag{4}$$

Therefore

$$||y(0)||^2 \leq \exp(2\mathbf{\Lambda_0} t)||y(t)||^2. \tag{5}$$

This is estimate is sharp, the energy version of the Fourier series representation.

Inverse design as an optimal control problem

We can rewrite the problem of inverse design as optimal control problem, consisting on the minimization of the functional

$$J(u^0) = \frac{1}{2}||u(T) - u^d||^2$$

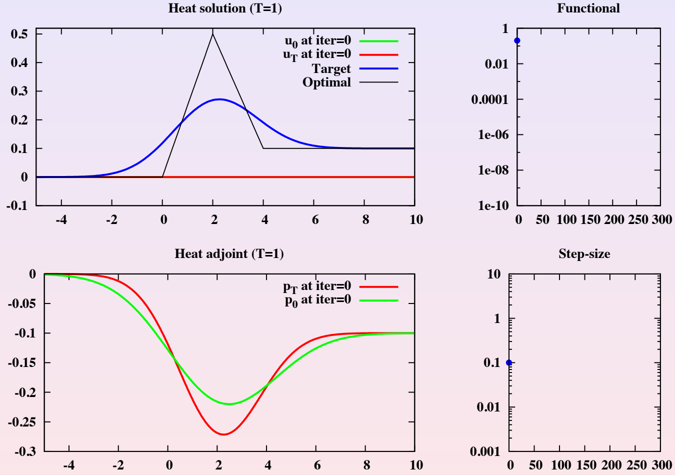
associated to the solutions of the forward equation

$$\begin{cases} u_t(t) + A(u(t)) = 0 \\ u(0) = u^0 \end{cases}$$

The minimization problem above can be proved to have at least a solution for a large class of models, targets and within reasonable classes of data.

It can be viewed as a **shooting problem**. Find u_0 so that $u(T) \sim u^d$.

The difficulty of finding a pick for the linear heat equation



A linear example

Most linear systems enjoy the property of backward uniqueness. There are however some exceptions:

$$f_t + f_x = 0;\ 0 < x < 1,\ t > 0; f(0,t) = 0,\ t > 0.$$

Then,

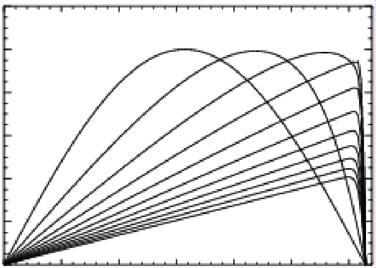
$$f(x,t) \equiv 0,\ \forall t \geq 1,\ 0 < x < 1.$$

This fact is closely linked to the theory of perfect and transparent boundary conditions for waves, PML,...

Irreversibility by nonlinearity

For hyperbolic Burgers equation (and more generally for first order non-linear conservation laws) irreversibility occurs because of shock formation:

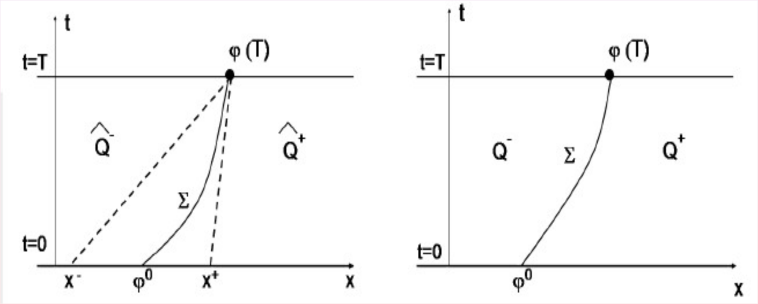
$$u_t + \partial_x [u^2] = 0$$



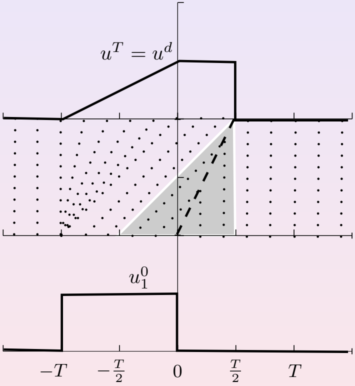
Solution as a pair: flow+shock variables

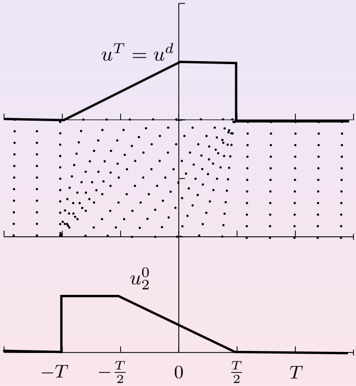
Then the pair (u, φ)= (flow solution, shock location) solves:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, & \text{in } Q^- \cup Q^+, \\ \varphi'(t) = \dfrac{(u^+(\varphi(t), t) + u^-(\varphi(t), t))}{2}, & t \in (0, T), \\ \varphi(0) = \varphi^0, & \\ u(x, 0) = u^0(x), & \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}. \end{cases}$$



Lack of backward uniqueness





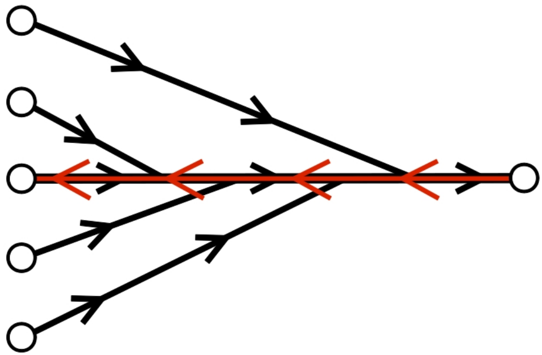
Numerical inverse design

Two issues:

- Build efficient numerical solvers to find one inverse design;
- Recover all the possible inverse designs.

All forward solutions are entropic ones

Only one out of all of the backward trajectories is entropic



Conservative schemes

Let us consider now numerical approximation schemes for the inviscid problem :

$$\begin{cases} u_j^{n+1} = u_n^j - \dfrac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), & j \in \mathbf{Z}, \mathbf{n} > \mathbf{0}. \\ u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbf{Z}. \end{cases}$$

The approximated solution u_Δ is given by

$$u_\Delta(t,x) = u_j^n, \quad x_{j-1/2} < x < x_{j+1/2},\ t_n \leq t < t_{n+1},$$

where $t_n = n\Delta t$ and $x_{j+1/2} = (j + \frac{1}{2})\Delta x$.

Is the large tine dynamics of these discrete systems, a discrete version of the continuous one?

3-point conservative schemes

1. Lax-Friedrichs

$$g^{LF}(u,v) = \frac{u^2+v^2}{4} - \frac{\Delta x}{\Delta t}\left(\frac{v-u}{2}\right),$$

2. Engquist-Osher

$$g^{EO}(u,v) = \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},$$

3. Godunov

$$g^{G}(u,v) = \begin{cases} \min\limits_{w\in[u,v]} \frac{w^2}{2}, & \text{if } u \le v, \\ \max\limits_{w\in[v,u]} \frac{w^2}{2}, & \text{if } v \le u. \end{cases}$$

Properties

These three schemes are well-known to satisfy the following properties:

- They converge to the entropy solution
- They are monotonic
- They preserve the total mass of solutions
- They are OSLC consistent:

$$\frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} \leq \frac{2}{n\Delta t}$$

- $L^1 \to L^\infty$ decay with a rate $O(t^{-1/2})$
- Similarly they verify uniform BV_{loc} estimates

Asymptotic correctness as $t \to \infty$?

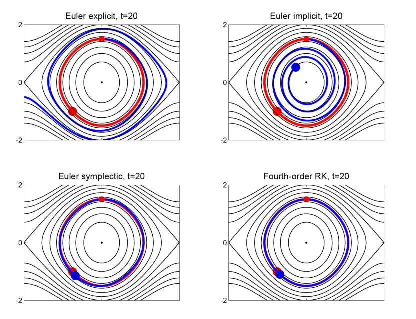
- All these methods converge in the classical sense of numerical analysis.
- This refers to convergence in finite time intervals $[0, T]$!!!
- But do they behave correctly as $t \to \infty$?
- Note that, computationally, roughly, you can choose Δx and Δt, but once you do this, you have to rely on what simulations give as $t \to \infty$. And this is relevant when solving numerically the minimisation problem since a gradient descent method requires the iterative resolution of the forward and adjoint dynamics many times, producing then the effect of solving the PDE over long time intervals.

Geometric integration

In the mathematical field of numerical ordinary differential equations, a geometric integrator is a numerical method that preserves geometric properties of the exact flow of a differential equation.

Hairer, Ernst; Lubich, Christian; Wanner, Gerhard (2002). Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations. Springer-Verlag.

https://en.wikipedia.org/wiki/Geometric_integrator



Joint work with L. Ignat & A. Pozo, Math of Computation, 2014

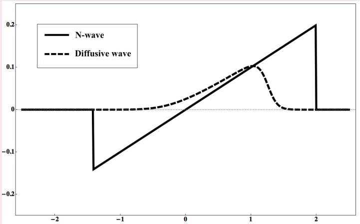
Consider the 1-D conservation law with or without viscosity:

$$u_t + \left[u^2\right]_x = \varepsilon u_{xx}, x \in \mathbb{R}, t > 0.$$

Then[1] :

- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

u_M is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass M of u_0), while N is the so-called hyperbolic N-wave.



[1] Y. J. Kim & A. E. Tzavaras, *Diffusive N-Waves and Metastability in the Burgers Equation*, SIAM J. Math. Anal. **33**(3) (2001), 607–633.

Conservative schemes

Let us consider now numerical approximation schemes

$$\begin{cases} u_j^{n+1} = u_n^j - \dfrac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), & j \in \mathbf{Z}, \mathbf{n} > \mathbf{0}. \\ u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbf{Z}, \end{cases}$$

The approximated solution u_Δ is given by

$$u_\Delta(t,x) = u_j^n, \quad x_{j-1/2} < x < x_{j+1/2},\ t_n \leq t < t_{n+1},$$

where $t_n = n\Delta t$ and $x_{j+1/2} = (j + \frac{1}{2})\Delta x$.

Is the large tine dynamics of these discrete systems, a discrete version of the continuous one?

3-point conservative schemes

1. Lax-Friedrichs

$$g^{LF}(u,v) = \frac{u^2+v^2}{4} - \frac{\Delta x}{\Delta t}\left(\frac{v-u}{2}\right),$$

2. Engquist-Osher

$$g^{EO}(u,v) = \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},$$

3. Godunov

$$g^{G}(u,v) = \begin{cases} \min\limits_{w\in[u,v]} \frac{w^2}{2}, & \text{if } u \le v, \\ \max\limits_{w\in[v,u]} \frac{w^2}{2}, & \text{if } v \le u. \end{cases}$$

Numerical viscosity

We can rewrite three-point monotone schemes in the form

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}=R(u_j^n,u_{j+1}^n)-R(u_{j-1}^n,u_j^n)$$

where the numerical viscosity R can be defined in a unique manner as

$$R(u,v)=\frac{Q(u,v)(v-u)}{2}=\frac{\lambda}{2}\Big(\frac{u^2}{2}+\frac{v^2}{2}-2g(u,v)\Big).$$

For instance:

$$R^{LF}(u,v)=\frac{v-u}{2\Delta t},$$

$$R^{EO}(u,v)=\frac{\lambda}{4}(v|v|-u|u|),$$

$$R^{G}(u,v)=\begin{cases}\frac{\lambda}{4}\mathrm{sign}(|u|-|v|)(v^2-u^2), & v\le 0\le u,\\ \frac{\lambda}{4}(v|v|-u|u|), & \text{elsewhere.}\end{cases}$$

Properties

These three schemes are well-known to satisfy the following properties:

- They converge to the entropy solution
- They are monotonic
- They preserve the total mass of solutions
- They are OSLC consistent:

$$\frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} \leq \frac{2}{n\Delta t}$$

- $L^1 \to L^\infty$ decay with a rate $O(t^{-1/2})$
- Similarly they verify uniform BV_{loc} estimates

Theorem (Lax-Friedrichs scheme)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \left| u^n \right|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solution u_Δ given by the Lax-Friedrichs scheme satisfies

$$\lim_{t \to \infty} t^{\frac{1}{2}(1-\frac{1}{p})} \left| u_\Delta(t) - w(t) \right|_{L^p(\mathbb{R})} = 0,$$

where the profile $w = w_{M_\Delta}$ is the unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2} \right)_x = \frac{(\Delta x)^2}{2\Delta t} w_{xx}, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$.

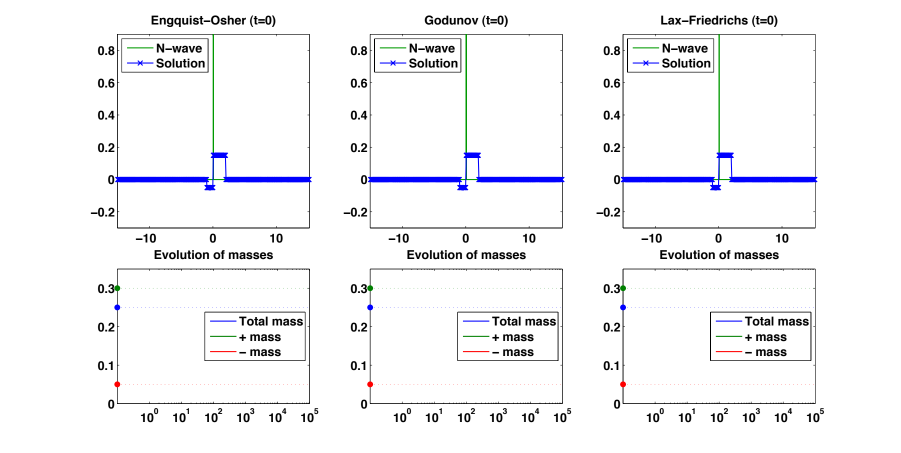
Theorem (Engquist-Osher and Godunov schemes)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \Big| u^n \Big|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solutions u_Δ given by Engquist-Osher and Godunov schemes satisfy the same asymptotic behavior but for the hyperbolic $N - wave$ $w = w_{p_\Delta, q_\Delta}$ unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, & \lim\limits_{t \to 0} \displaystyle\int_0^x w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and
$p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^x u_\Delta^0(z)dz$ and $q_\Delta = \max_{x \in \mathbb{R}} \int_x^\infty u_\Delta^0(z)dz.$

Example


Engquist–Osher (t=0)
Godunov (t=0)
Lax–Friedrichs (t=0)
N–wave
Solution
0.8
0.6
0.4
0.2
0
–0.2
–10
0
10
Evolution of masses
Total mass
+ mass
– mass
0.3
0.2
0.1
0
10^0 10^1 10^2 10^3 10^4 10^5

Similarity variables

Let us consider the change of variables given by:

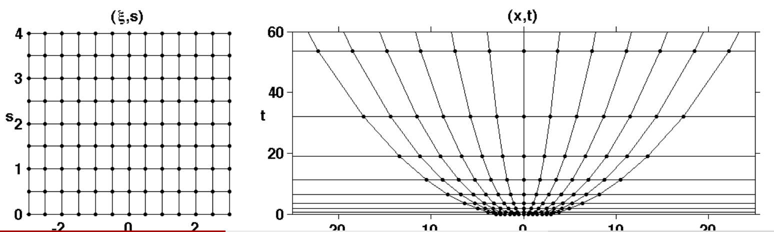
$$s = \ln(t+1), \qquad \xi = x/\sqrt{t+1}, \qquad w(\xi, s) = \sqrt{t+1}\; u(x,t),$$

which turns the continuous Burgers equation into

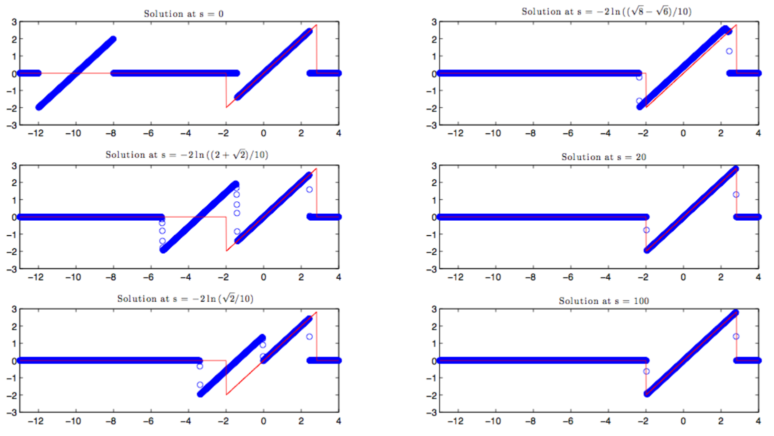
$$w_s + \left(\frac{1}{2}w^2 - \frac{1}{2}\xi w\right)_\xi = 0, \quad \xi \in \mathbf{R}, s > 0.$$

The asymptotic profile of the N-wave becomes a steady-state solution:

$$N_{p,q}(\xi) = \begin{cases} \xi, & -\sqrt{2p} < \xi < \sqrt{2q}, \\ 0, & \text{elsewhere}, \end{cases}$$



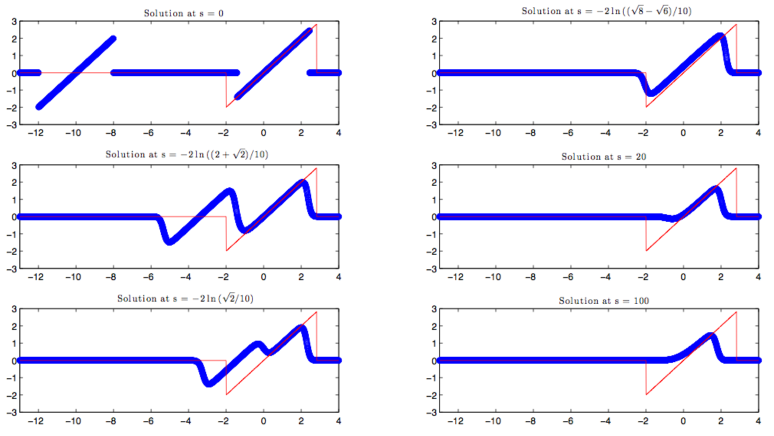
Examples



Convergence of the numerical solution using Engquist-Osher scheme (circle dots) to the asymptotic N-wave (solid line). We take $\Delta\xi = 0.01$ and $\Delta s = 0.0005$.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

Examples



Numerical solution using the Lax-Friedrichs scheme (circle dots), taking $\Delta\xi = 0.01$ and $\Delta s = 0.0005$. The N-wave (solid line) is not reached, as it converges to the diffusion wave.

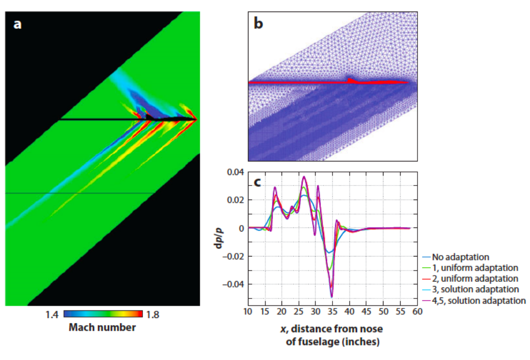
Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

Conclusion

- Numerical viscosity needs to be tuned carefully to ensure that the asymptotic behavior as $t \to \infty$ of numerical solutions coincides with that of the PDE solutions.
- Similar issues have been considered in the context of linear wave equation [2]. In that frame, it is by now well known that numerical viscosity needs to be added to ensure an exponential decay rate, informs with respect to the mesh parameters, but the issue of recovering, precisely, the decay rate of the congruous PDE solutions have not been considered so far:

[2] K. Ramdani, T. Takahashi, M. Tucsnak, Uniformly exponentially stable approximations for a class of second order evolution equations. Application to LQR problems. ESAIM COCV,13 (2007), no. 3, 503-527.

The problem of inverse design, motivated by the problem of sonic-boom, and more precisely by the determination of the profile of the initial signature so to make sure it is acceptable when reaching earth, according to present regulations, can be formulated as an optimization or control problem in which the initial datum of the PDE under consideration.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012. 44:505 – 26.

Inverse design as an optimal control problem

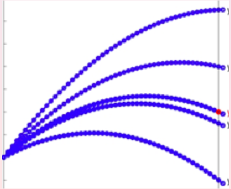
We rewrite the problem of inverse design and as optimal control problem, consisting on the minimization of the functional

$$J(u^0) = \frac{1}{2} \int_{-\infty}^{\infty} |u(x, T) - u^d(x)|^2 dx.$$

associated to the solutions of the Burgers equation

$$\begin{cases} \partial_t u + \partial_x (u^2) - \varepsilon u_{xx} = 0 \\ u(x, 0) = u^0(x). \end{cases}$$

The minimization problem above can be proved to have a solution for a large class of targets and within reasonable classes of initial data.
It can be viewed as a **shooting problem**. Find u_0 so that $u(T) \sim u^d$.



The problem of the identification of the initial datum has many other potential applications and, accordingly, has been considered previously:

- G. Haine, Observateurs en dimension infinie. Application à l'etude de quelques problèmes inverses, PhD Thesis, Université de Lorraine, 2013.
- Y. Li, S. Osher, and R. Tsai, *Heat source identification based on* l_1 *constrained minimization*, Inverse Probl. and Imaging **8** (2014), no. 1, 199–221.

Continuous inverse design: The wave equation

Consider:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

Given ω, an open subset of Ω, it is by now well known that

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$

holds in time $T > 0$ for some $C > 0$ provided the so-called Geometric Control Condition (GCC) by Ralston and Bardos-Lebeau-Rauch holds.[3]

[3] The GCC requires, roughly, that all rays of Geometric Optics propagating in Ω and bouncing on the boundary get to the observation subset ω in time T.

Continuous inverse design: The heat equation

Consider:

$$\begin{cases} y_t - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x, 0) = y^0(x), & \text{in} \quad \Omega. \end{cases}$$

Given any open subset ω of Ω, it is by now well known[4] that

$$\int_0^T \int_\Omega e^{-A/t} |y|^2 dxdt \leq C \int_0^T \int_\omega |y|^2 dxdt.$$

holds in an arbitrarily short time $T > 0$ for some $A > 0$, $C > 0$ without any further geometric restriction.

This can also be re-written in terms of the energy of the initial data recovered:

$$\sum_k e^{-B\sqrt{\lambda_k}} |\hat{y}_k^0|^2 \leq C \int_0^T \int_\omega |y|^2 dxdt.$$

[4] Fursikov and Imanuvilov (1996)

Time reversible versus time irreversible systems

- Wave equations are time-reversible. Initial data can be fully reconstructed provided enough information on solutions is given (along bicharacteristics).
- Heat equations are strongly time-irreversible. Initial data can be only reconstructed in an exponentially weak (in the Fourier sense) norm. This makes reconstruction algorithms to be typically ill-posed.

Ill-posed in the sense of Hadamard

The mathematical term **well-posed problem** stems from a definition given by Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that

1. A solution exists
2. The solution is unique
3. The solution's behavior changes continuously with the initial conditions.

$$\sum_k e^{-B\sqrt{\lambda_k}} |\hat{y}_k^0|^2 \leq C \int_0^T \int_\omega |y|^2 dx dt.$$

Links: the Kannai transform

The **Kannai transform** allows transferring the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} W(s) ds$$

where $W(x,s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

$$W_{ss} + AW = 0 \quad + \quad K_t - K_{ss} = 0 \quad \to \quad U_t + AU = 0,$$

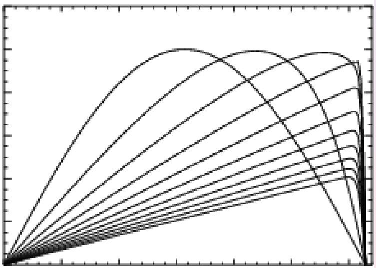
$$W_{ss} + AW = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \to \quad iU_t + AU = 0.$$

We refer to the more recent paper by S. Ervedoza and E. Z., ARMA 2013, for a reverse transformation.

Irreversibility by nonlinearity

Note however that the viscous Burgers equation under consideration is a viscous version of the hyperbolic Burgers equation for which shock discontinuities arise:

$$u_t + \partial_x [u^2] = 0.$$



Consider again the problem of minimizing

$$J(u^0) = \frac{1}{2} \int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

subject to the Burgers equation

$$\partial_t u + \partial_x(u^2) - \varepsilon u_{xx} = 0; u(x,0) = u^0(x).$$

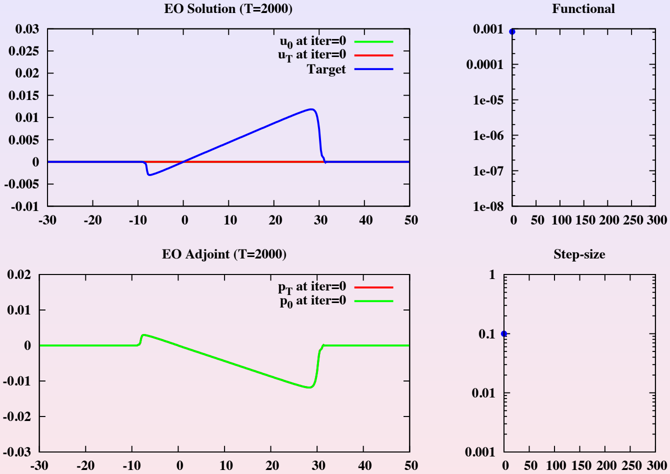
The **discrete version** of the functional:

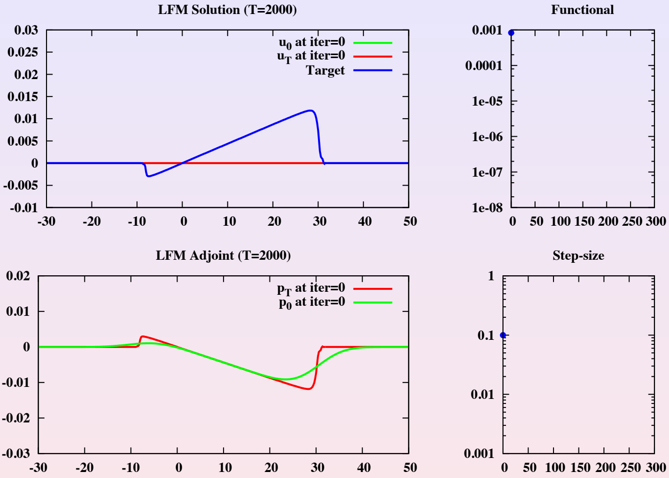
$$J^\Delta(u_\Delta^0) = \frac{\Delta x}{2} \sum_{j=-\infty}^{\infty} (u_j^{N+1} - u_j^d)^2,$$

where $u_\Delta = \{u_j^k\}$ solves a numerical discretization of the PDE based on some of the conservative schemes for conservation laws mentioned above

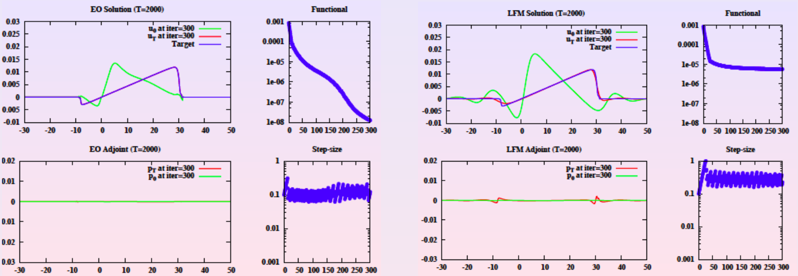
$$u_j^{n+1} = u_n^j - \frac{\Delta t}{\Delta x} \left(g_{j+1/2}^n - g_{j-1/2}^n \right).$$

In view of the very different asymptotic behavior of numerical solutions in large times, we expect a different performance of the discrete optimization achieved.
In fact, we expect Engquist-Osher to perform well, but Lax-Friedrisch to have difficulties to recover the correct inverse design.


EO Solution (T=2000)
u0 at iter=0
uT at iter=0
Target
Functional
EO Adjoint (T=2000)
pT at iter=0
p0 at iter=0
Step-size

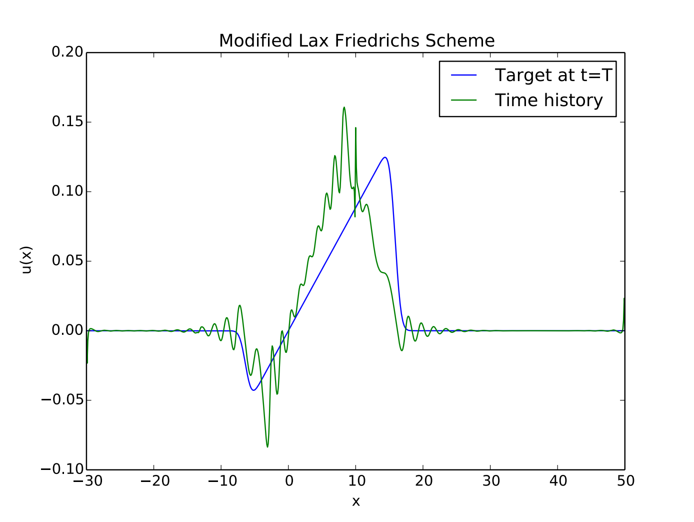


Is the iterative algorithm trapped in a local minimizer?

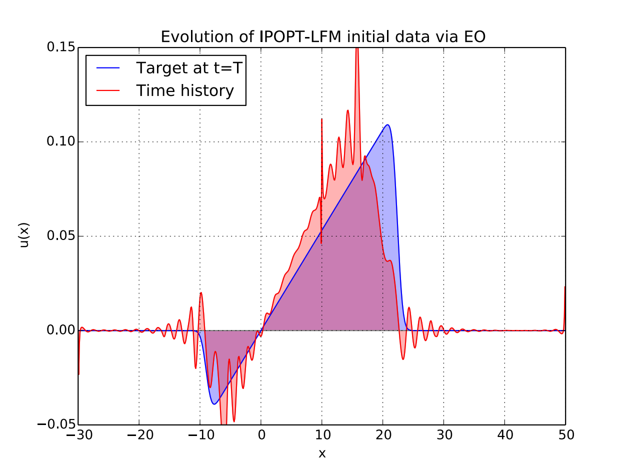


Global performance of the numerical optimization with EO (left) versus MLF (right)

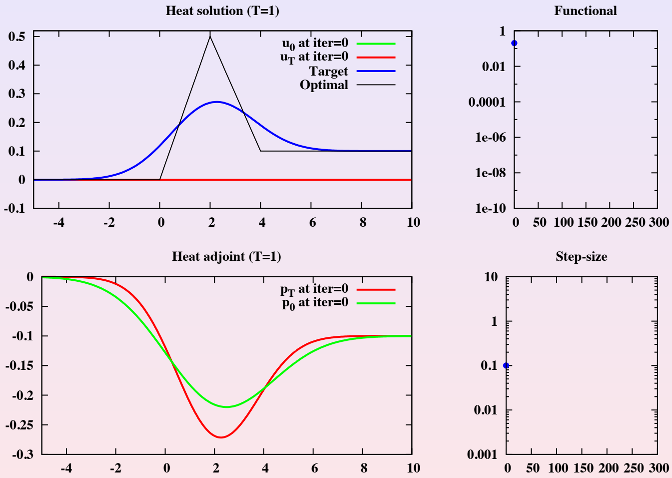
This is what the IPOPT software do (N. Allihverdi)



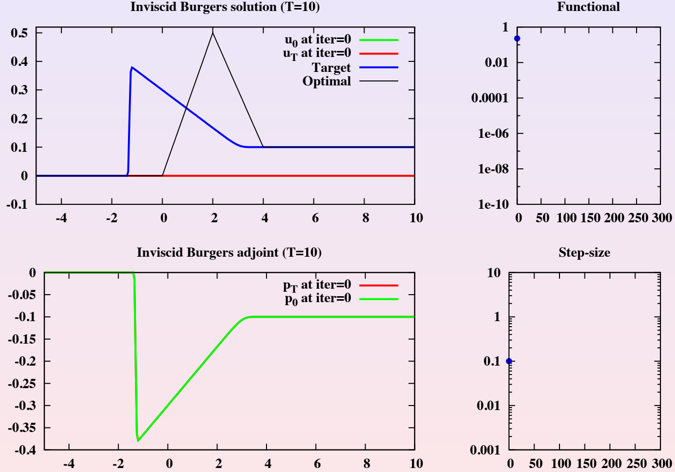
And this is the **bad** performance of the recovered initial profile when employing EO dynamics.



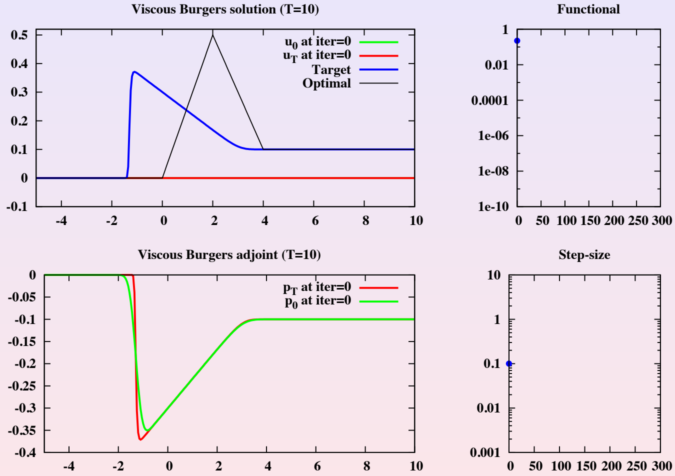
The difficulty of finding a pick for the linear heat equation



The obstacle of the lack of backward uniqueness for hyperbolic conservation laws



The superposition of both phenomena for viscous Burgers

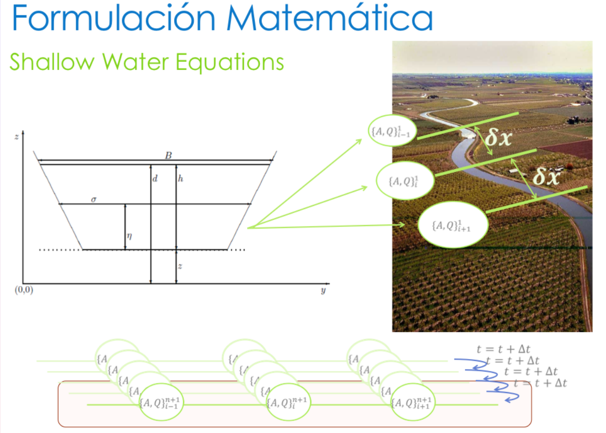


Adaptivity allows recovering similar results much faster!

	CPU TIME variable dt (s)	CPU TIME constant dt
ROE UPWIND	2.01	58.36
LAX-FRIEDRICHS	2.35	68.39
LAX-WENDROFF	2.51	70.46
ENGQUIST-OSHER	1.98	57.81

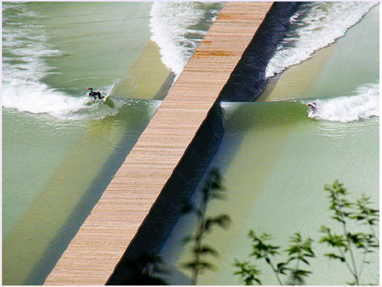
Results by Mario Morales using time adaptivity

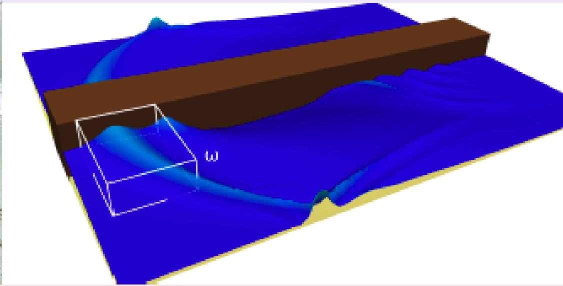
Water management



Joint research direction with Zaragoza team leaded by Pilar Garcia-Navarro (Mario Morales post-doc)

Wave maker

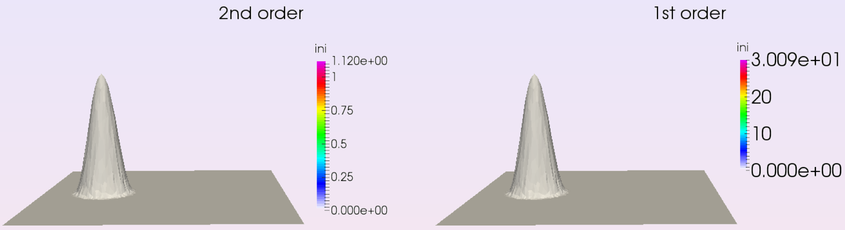


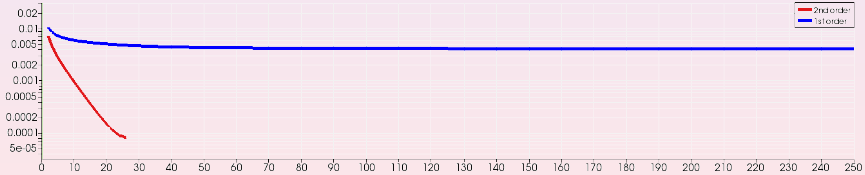


Instant Sport Wave Garden (`www.wavegarden.com` and Robin Goix' Ecole Polytechnique master within NUMERIWAVES team. [5]

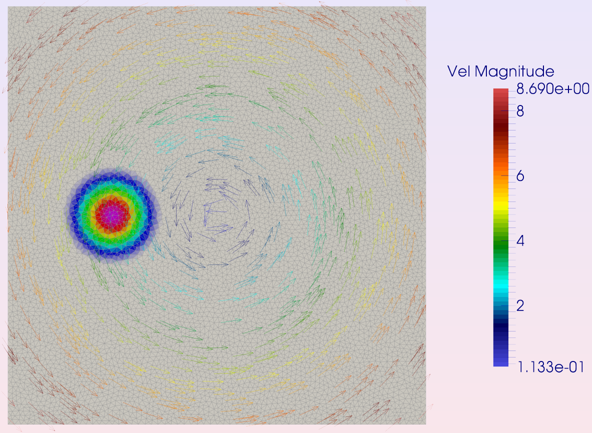
[5] H. Nersisyan, D. Dutykh and E. Z. Generation of two-dimensional water waves by moving bottom disturbances, IMA J. Appl. Math., (2015) 80, 1235Ð1253.

M. Morales, inverse design for a 2-d linear turning transport model





M. Morales, inverse design for a 2-d linear turning transport model



J. Murillo, P. Garcia-Navarro and J. Burguete, Analysis of a second-order upwind method for the simulation of solute transport in 2D shallow water

Doswell frontogenesis, M. Morales & E. Z. [a]

[a] C.A. Doswell III, A kinematic analysis of frontogenesis associated with a nondivergent vortex Journal of the atmospheric sciences, 1984.

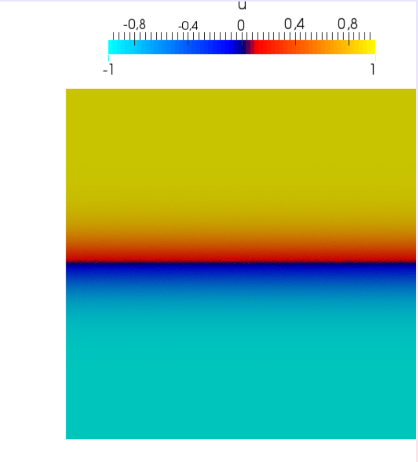
Frontogenesis is a meteorological process of tightening of horizontal temperature gradients to produce fronts.

$$\frac{\partial u}{\partial t} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0$$

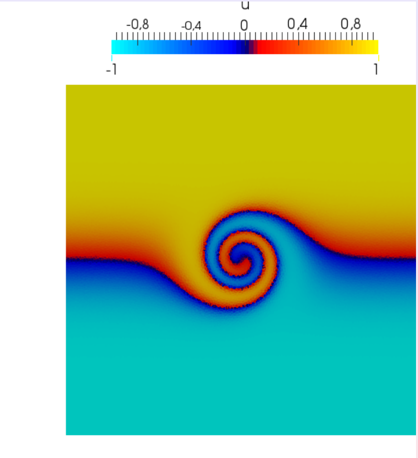
$$a(x,y) = -y\ f(r) \qquad b(x,y) = x\ f(r) \qquad f(r) = \frac{1}{r}v(r)$$

$$v(r) = \overline{v} sech^2(r)\tanh(r) \qquad r = \sqrt{x^2+y^2} \qquad \overline{v} = 2.59807$$

- Initial condition $\qquad u(x,y,0) = \tanh(y/delta)$
- $\delta =$ thickness of the front zone. $\delta \to 0 \sim$ sharp fronts.
- Exact solution $\qquad \boxed{u(x,y,t) = \tanh\left(\frac{y\cos(ft) - x\ \sin(ft)}{\delta}\right)}$



Initial condition



Exact solution

Conclusions

Lots to be done on:

- Numerical algorithms preserving large time asymptotics for nonlinear PDEs.
- Impact on inverse design.
- Local versus global minimizers.
- Fast solvers/adaptivity
- Multi-d
- Important applications.

All this needs to be made in a multidisciplinary environment so to assure impact on Engineering and Sciences

Thanks to coworkers: N. Allaverdhi, D. Dutykh, R. Goix, L. Ignat, A. Marica, M. Morales, H. Nersisyan, A. Pozo...