Control & numerics: Heat and Waves

Enrique Zuazua

Chair of Dynamics, Control and Numerics
Department of Data Science
FAU
Erlangen, Germany
https://caa-avh.nat.fau.eu

Some references:

- E. Z. "Controllability and Observability of Partial Differential Equations: Some results and open problems", in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, 2006, pp. 527-621.
- E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.
- S. Ervedoza and E. Z. The Wave Equation: Control and Numerics, in "Control and stabilization of PDE's", P. M. Cannarsa y J. M. Coron, eds., "Lecture Notes in Mathematics", CIME Subseries, Springer Verlag, to appear.
- A. Münch and E. Z. Numerical approximation of null controls for the heat equation through transmutation, J. Inverse Problems, 2010.

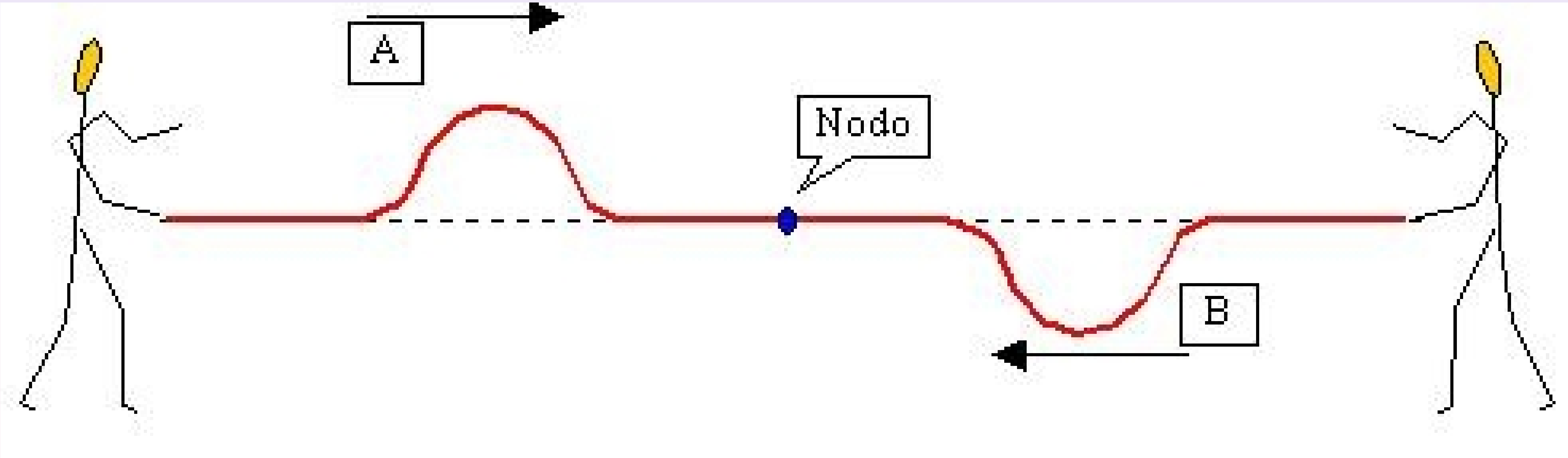
Control of $1-d$ vibrations of a string

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$


A
Nodo
B

The dual observation problem

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\, 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\, \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

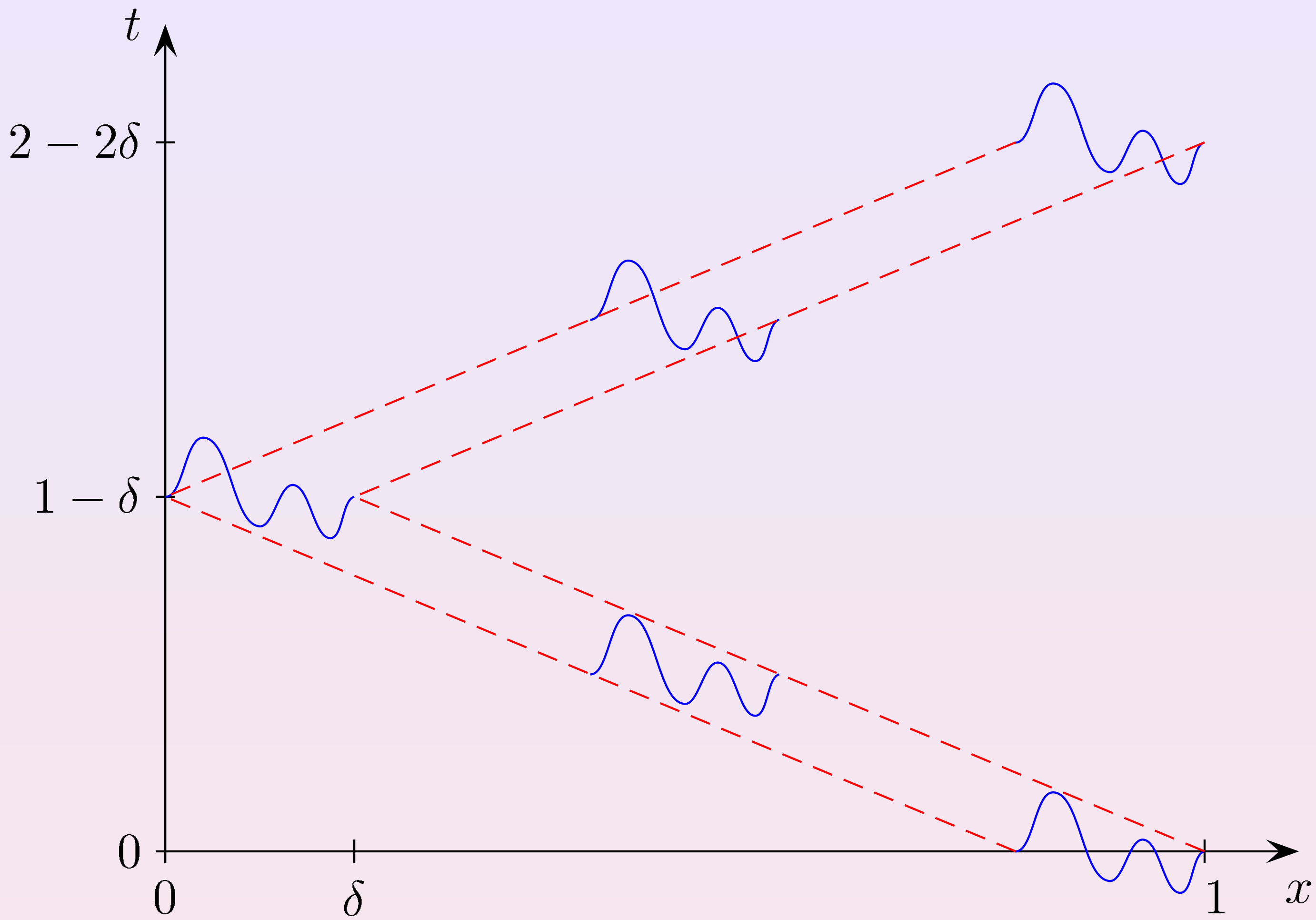
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2\right] dx = E(0), \ \ \forall 0 \le t \le T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \le C(T)\int_0^T |\varphi_x(1,t)|^2\, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

Construction of the Control

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1, t),$$

where u is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0, 1) \times L^2(0, 1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1, t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1},$$

in the space $H_0^1(0, 1) \times L^2(0, 1)$.
Note that J is convex. The continuity of J in $H_0^1(0, 1) \times L^2(0, 1)$ is guaranteed by the fact that $\varphi_x(1, t) \in L^2(0, T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY.

The continuous numerical approach: Gradient algorithms

The control was characterized as being the minimizer over $H_0^1(0,1) \times L^2(0,1)$ of

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1} .$$

We produce an algorithm in which:

- We replace J by some numerical approximation J_h with an order h^θ.
- We apply a gradient iteration algorithm to J_h.

The following holds:

Theorem

(S. Ervedoza & E. Z., 2011)
In

$$K \sim C|\log(h)|$$

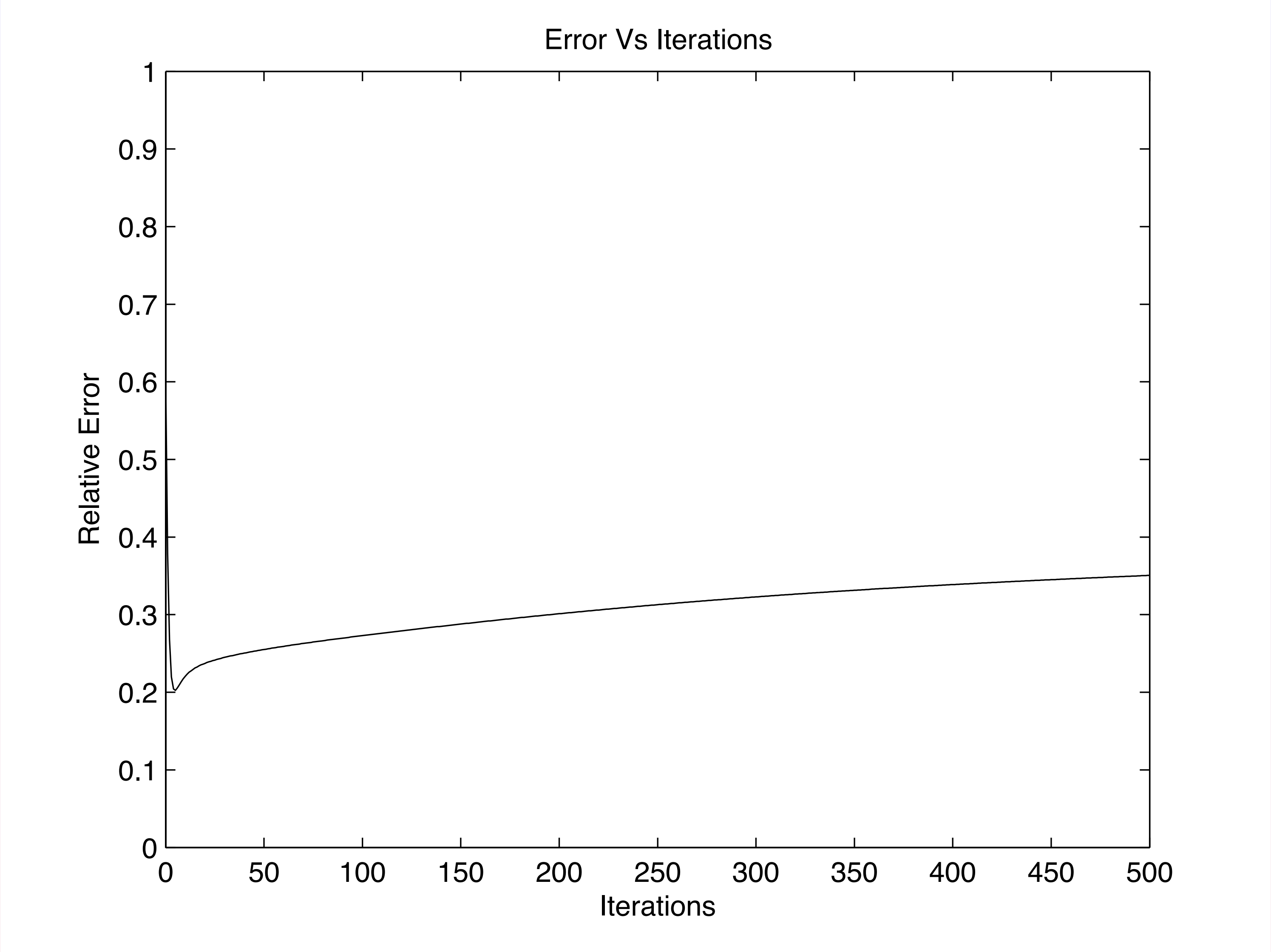
iterations, the controls v_h^K *obtained after applying* K *iterations of the gradient algorithm to* J_h *fulfill:*

$$||v - v_h^K|| \leq C|\log(h)|^{\max(\theta,1)} h^\theta.$$

Note that for the classical Finite Difference and Finite Element methods for the wave equation the convergence order is $\theta = 2/3$.

We have developed the continuous program successfully!

Note that the error estimate deteriorates if $K >> C|\log(h)|$!!!



... and, therefore, the method has to be used with much care since, after all, we are dealing with an **unstable, non-robust algorithm**....

But one might want to take a shortcut controlling a finite-dimensional reduced dynamics.

Set $h = 1/(N+1) > 0$ and consider the mesh

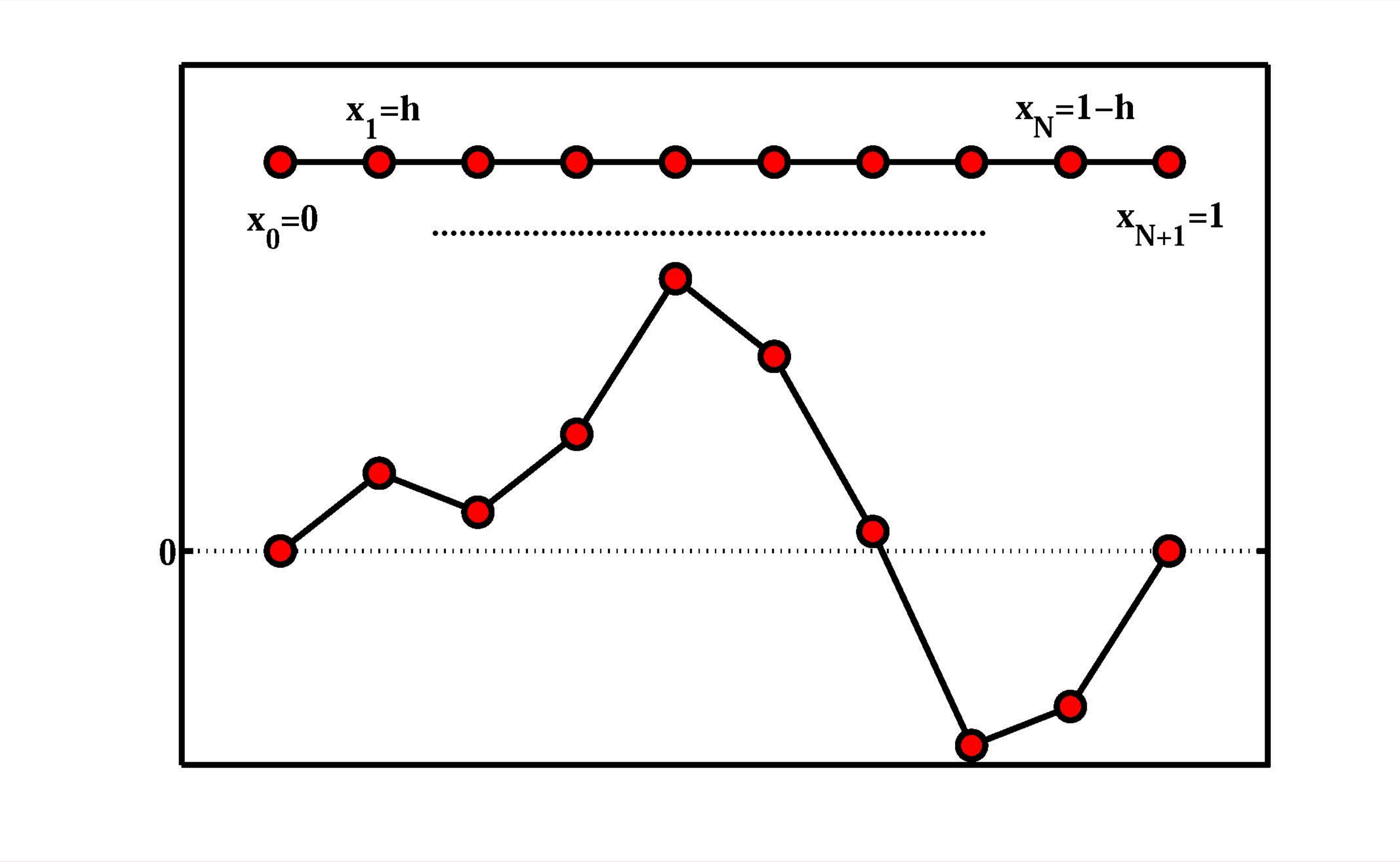
$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0,1]$ into $N+1$ subintervals

$$I_j = [x_j, x_{j+1}],\ j = 0, ..., N.$$

Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} \varphi_j'' - \frac{1}{h^2}\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ \varphi_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ \varphi_j(0) = \varphi_j^0,\ \varphi_j'(0) = \varphi_j^1, & j = 1, \dots, N. \end{cases}$$



From finite-dimensional dynamical systems to infinite-dimensional ones in purely conservative dynamics.....

Then it should be sufficient to minimize the discrete functional

$$J_h(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T \frac{|\varphi_N(1,t)|^2}{h^2} dt + h\sum_{j=1}^{N} \varphi_j^1 y_j^0 - h\sum_{j=1}^{N} \varphi_j^0 y_j^1,$$

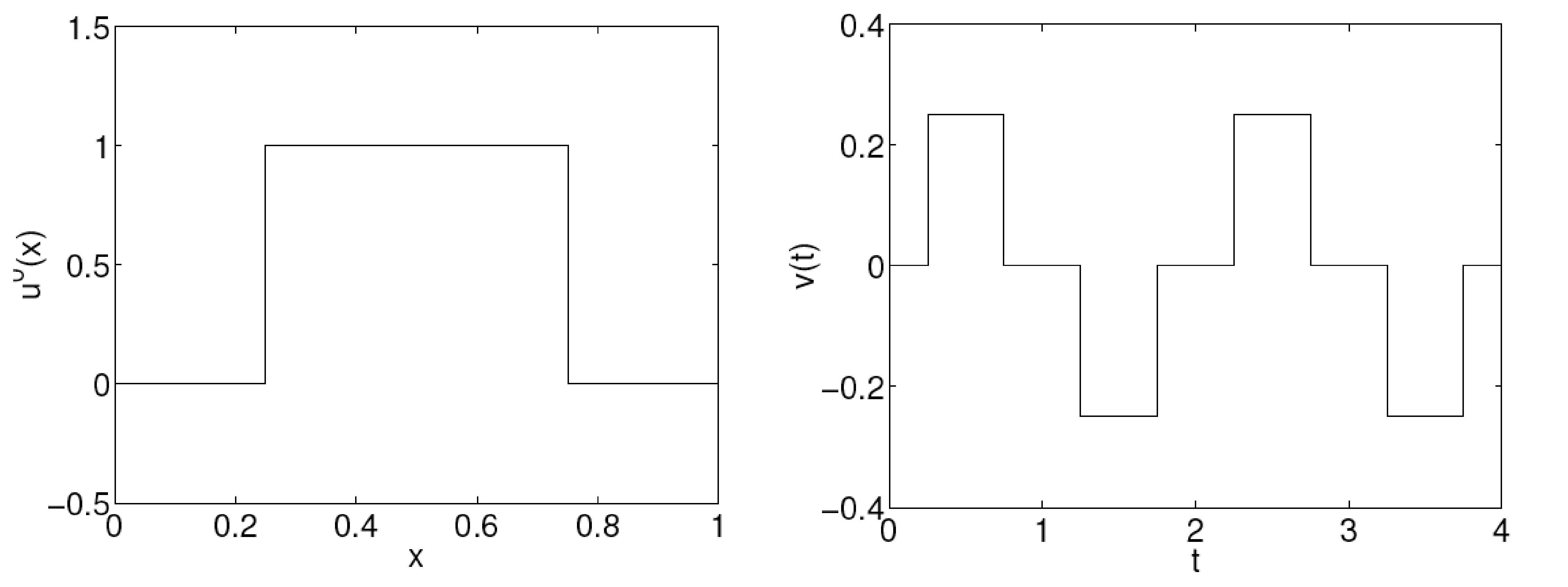
which is a discrete version of the functional J of the continuous wave equation since

$$-\frac{\varphi_N(t)}{h} = \frac{\varphi_{N+1} - \varphi_N(t)}{h} \sim \varphi_x(1,t).$$

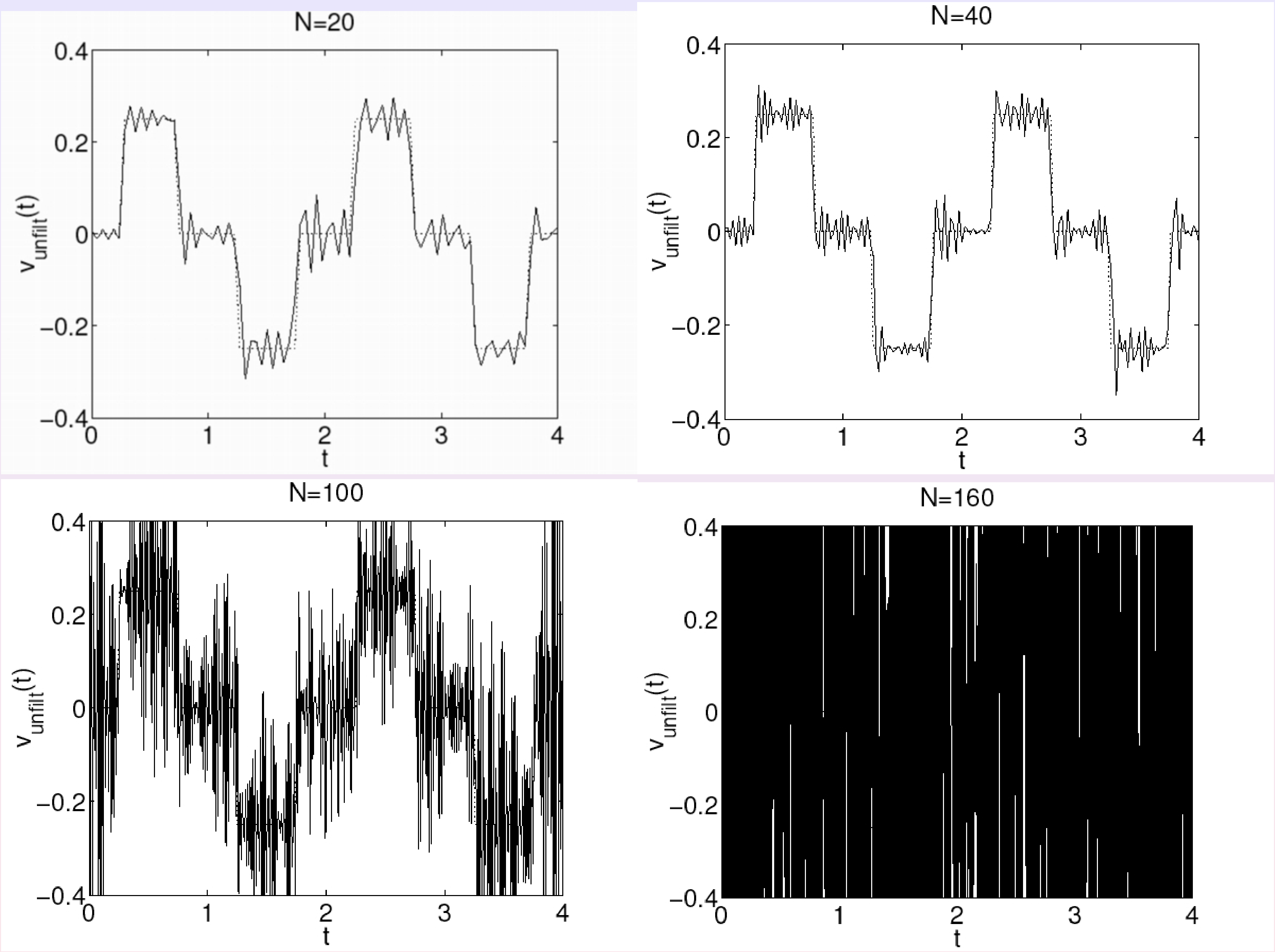
Then

$$v_h(t) = -\frac{\varphi_N^\star(t)}{h}.$$

A NUMERICAL EXPERIMENT



Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.
Plot of the time evolution of the exact control for the wave equation in time $T = 4$.



The control diverges as $h \to 0$.

WHY?

The Fourier series expansion shows the analogy between continuous and discrete dynamics.

Discrete solution:

$$\vec{\varphi} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h}\, t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h}\, t\right) \right) \vec{w}_k^h.$$

Continuous solution:

$$\varphi = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

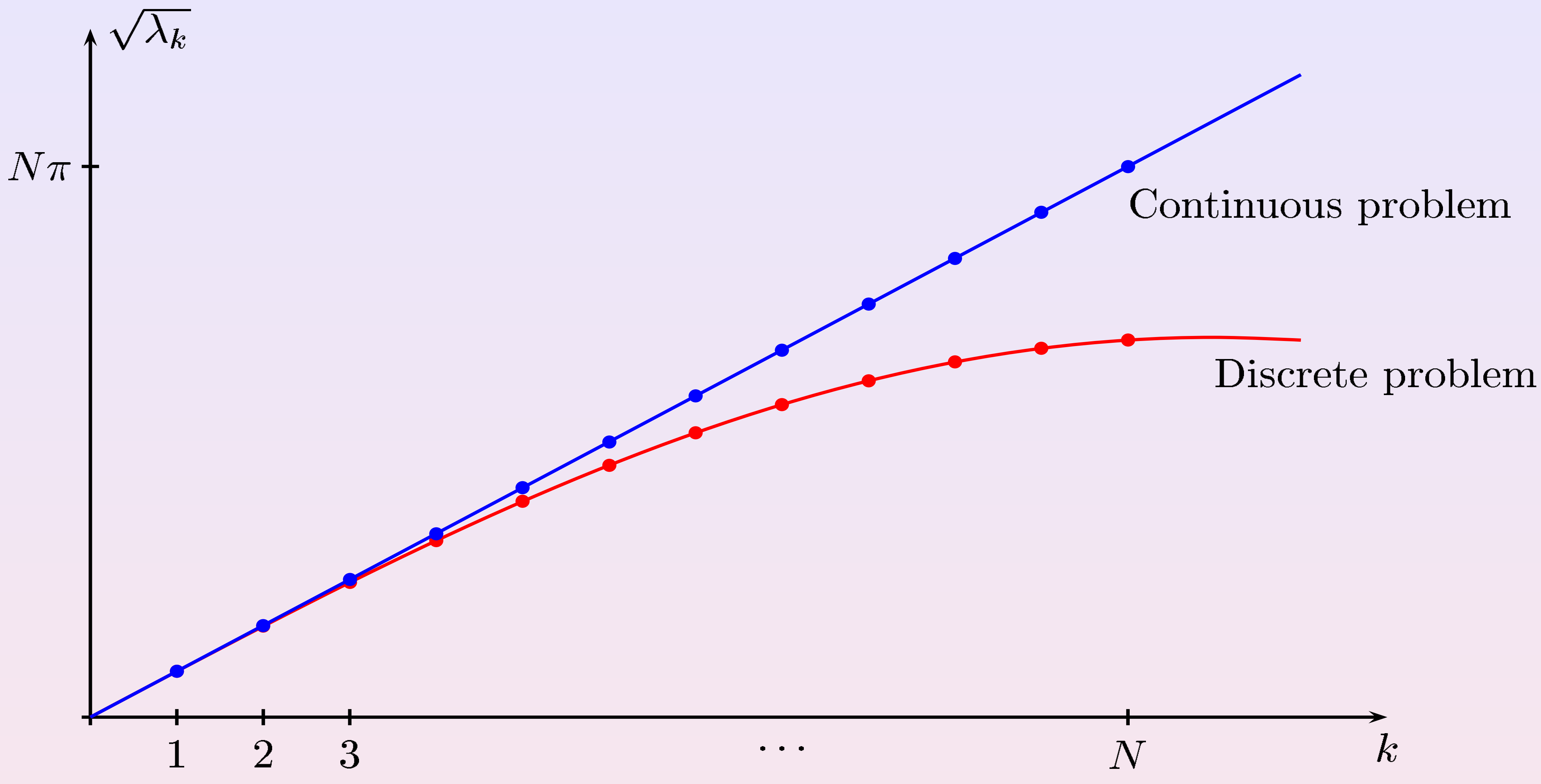
Recall that the discrete spectrum is as follows and converges to the continuous one:

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

$$w_k^h = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi jh),\ k, j = 1, \ldots, N.$$

The only relevant differences arise at the level of the dispersion properties and the group velocity. High frequency waves do not propagate, remain captured within the grid, without never reaching the boundary. This makes it impossible the uniform boundary control and observation of the discrete schemes as $h \to 0$.

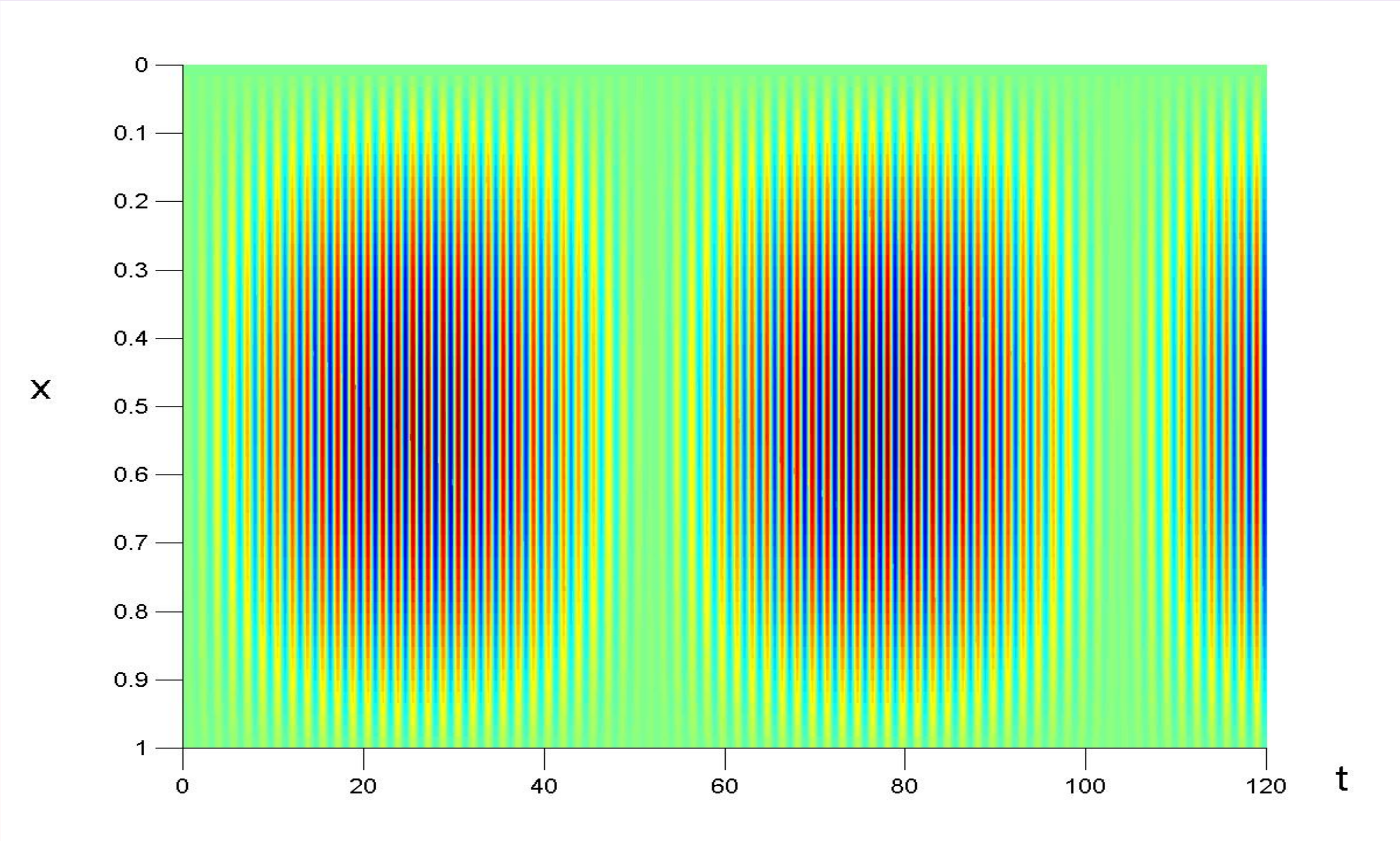


Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

A numerical phamtom

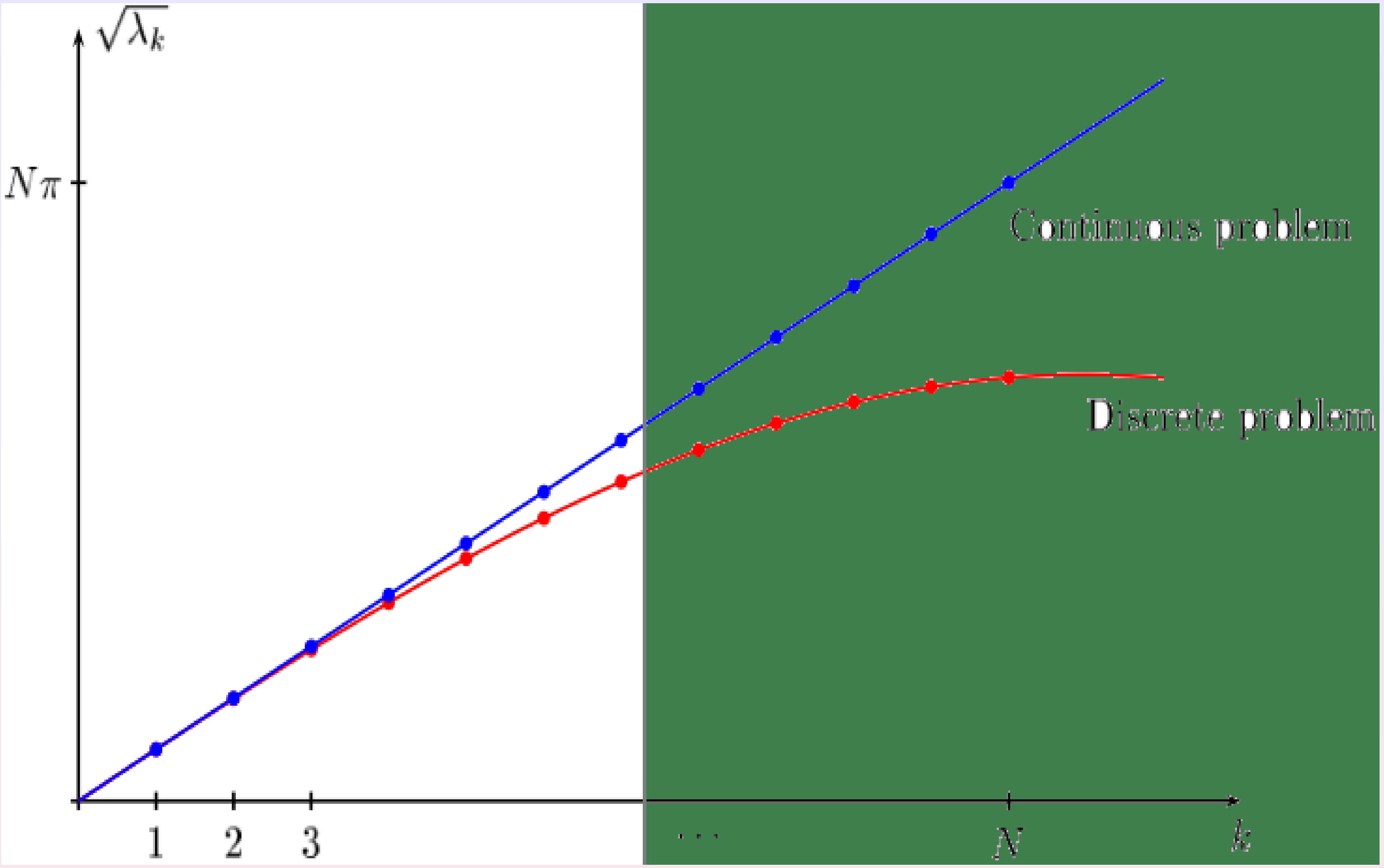
$$\vec{\varphi} = \exp\left(i\sqrt{\lambda_N(h)}\,t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\,t\right)\vec{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap: $\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h$.



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120$.

Fourier filtering



To filter the high frequencies, i.e. keep only the components of the solution corresponding to indexes: $k \leq \delta/h$ with $0 < \delta < 1$. This guarantees that the group velocity remains uniformly bounded below and allows observing uniformly filtered solutions in time $T(\delta) > 2$ such that $T(\delta) \to 2$ as $\delta \to 0$.

Relaxed controls:

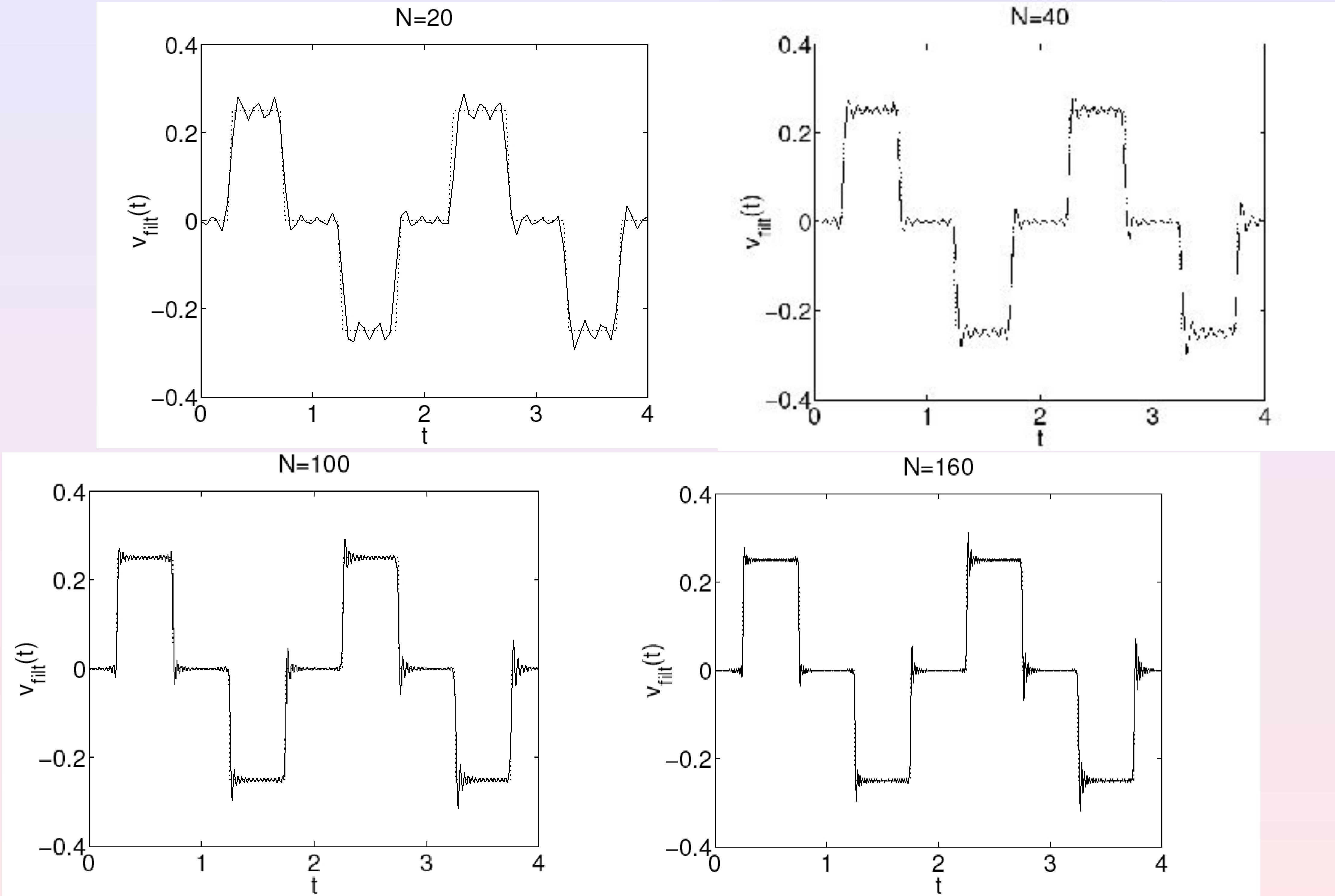
Then, the filtering algorithm can be implemented as follows:

- Minimize J_h over the class of filtered solutions with filtering parameter $0 < \delta < 1$ and $T > T(\delta)$;
- This yields controls v_h^δ such that
  - $v_h^\delta \to v$ as $h \to 0$;
  - The corresponding states $\vec{y}_h$ satisfiy:

$$\pi_\delta(\vec{y}_h) \equiv \pi_\delta(\vec{y}_h{}') \equiv 0.$$

This is a relaxed version of the controllability condition.

Numerical experiment, revisited, with filtering

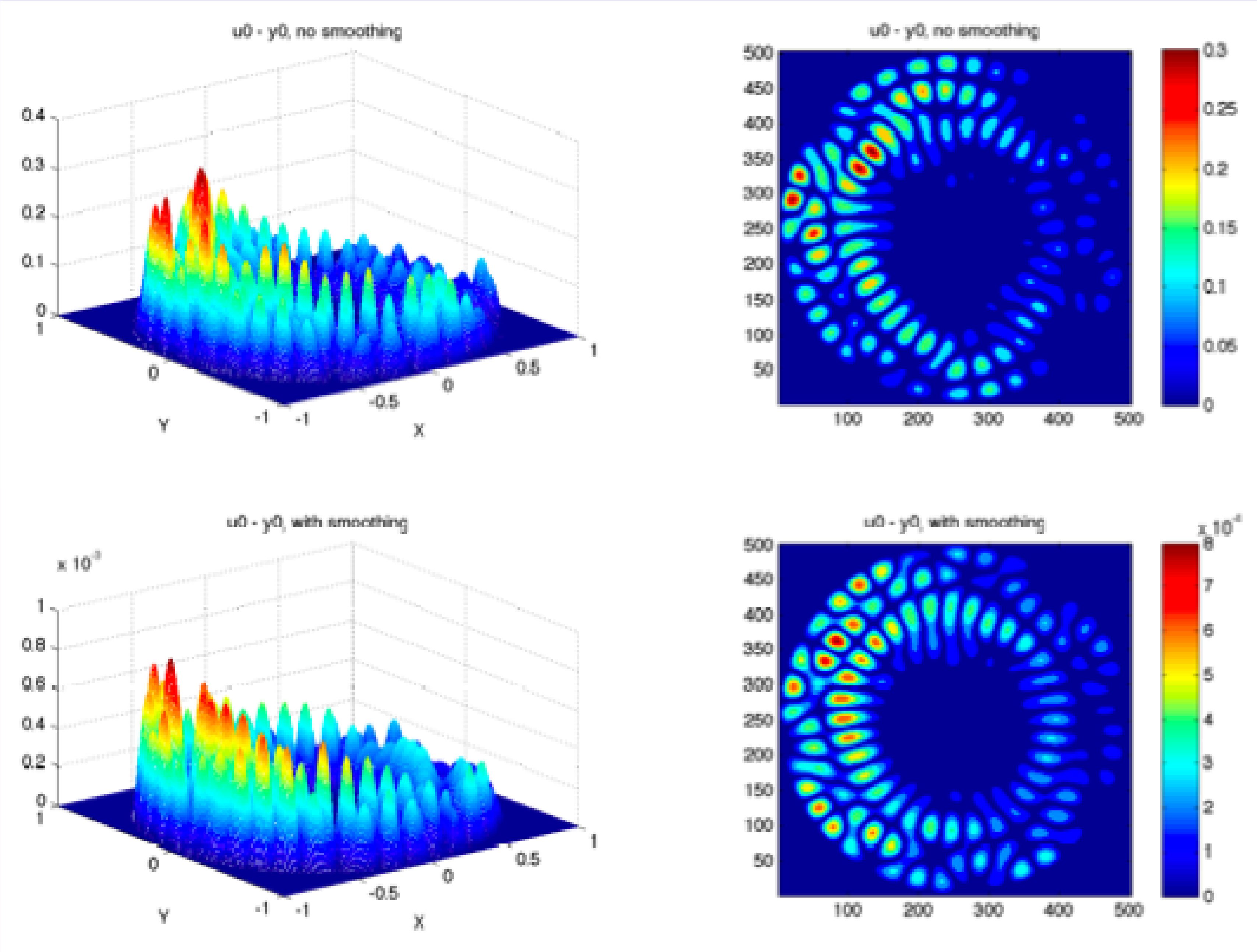


With appropriate filtering the control converges as $h \to 0$.

The discrete approach when applied directly fails, but it can cured borrowing ideas from the continuous analysis. The bonus is that:

- We compute numerical approximations of the controls that perform well, in an identified manner, controlling a Fourier projection of solutions at the discrete level.
- The algorithm converges is stable and robust, an the error diminishes as the number of iterations $\to \infty$.

Controls in multi-d may develop complex and unexpected patterns, in view of the laws of Geometric Optics.



G. Lebeau and M. Nodet, Experimental Study of the HUM Control Operator for Linear Waves, Experimental Mathematics, 2010.

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} y_t - \Delta u = v 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

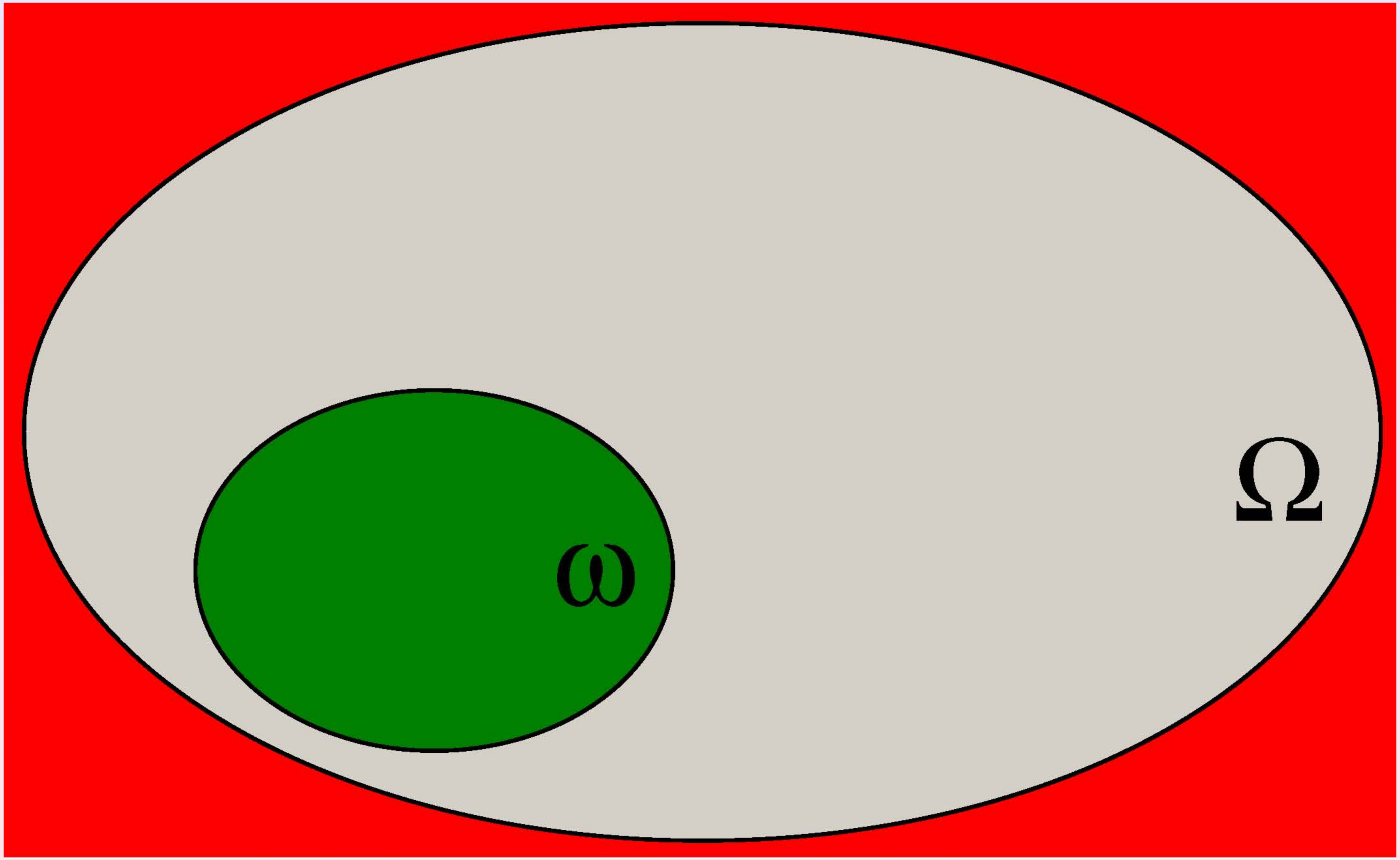
1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $y^0 \in L^2(\Omega)$ and $v \in L^2(Q)$ so that (4) admits an unique solution

$$y \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$y = y(x, t) = solution = state, \; v = v(x, t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function v.


ω
Ω

The model:

$$\begin{cases} y_t - \Delta y = v1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4}$$

Objective:

$$y(T) \equiv 0.$$

The control can be built as follows: Consider the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) y^0 dx. \quad (5)$$

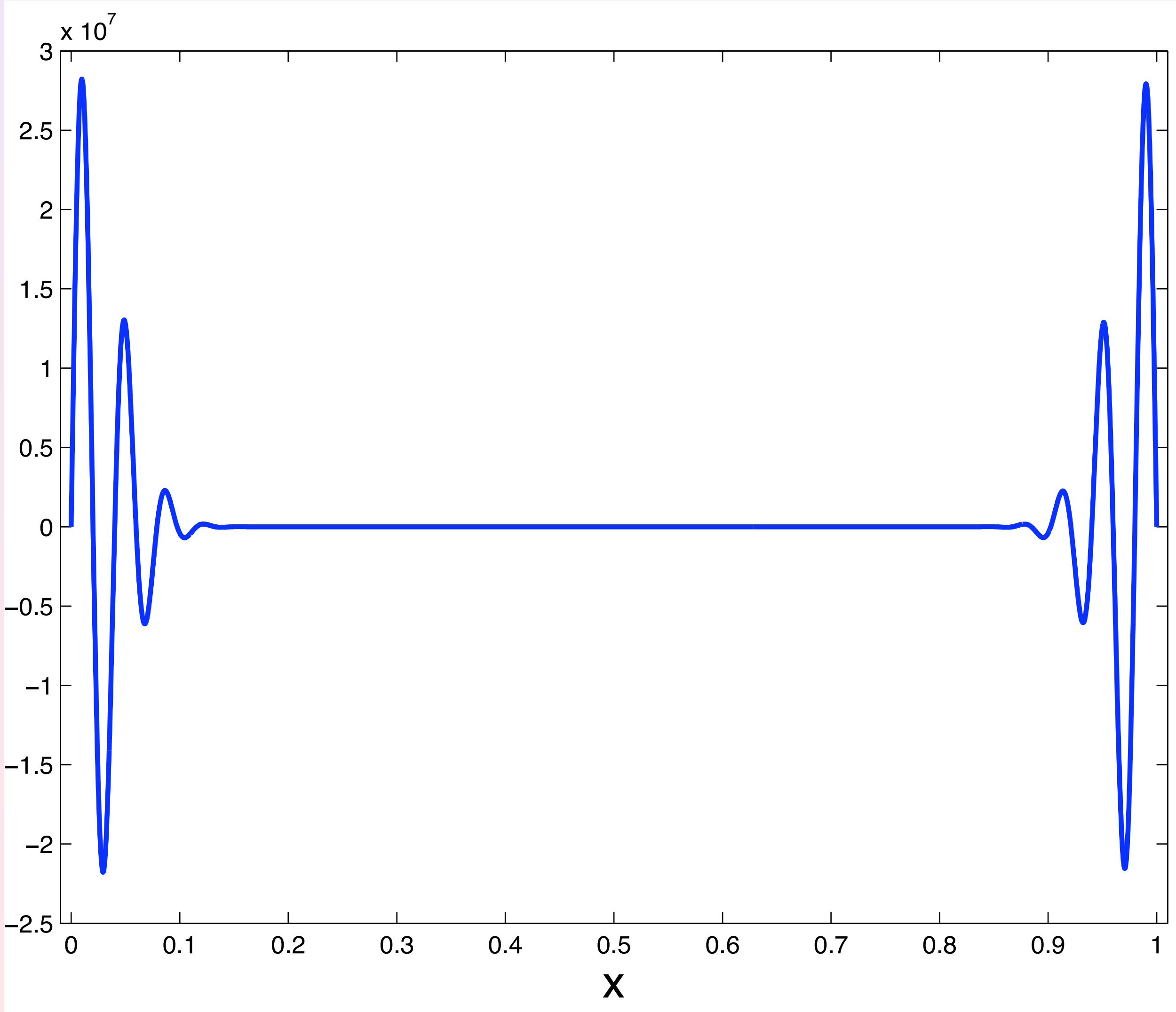
$J : L^2(\Omega) \to \mathbb{R}$ is continuous, and convex.
But, is it coercive?
If yes, the minimizer $\hat{\varphi}^0$ exists and the control

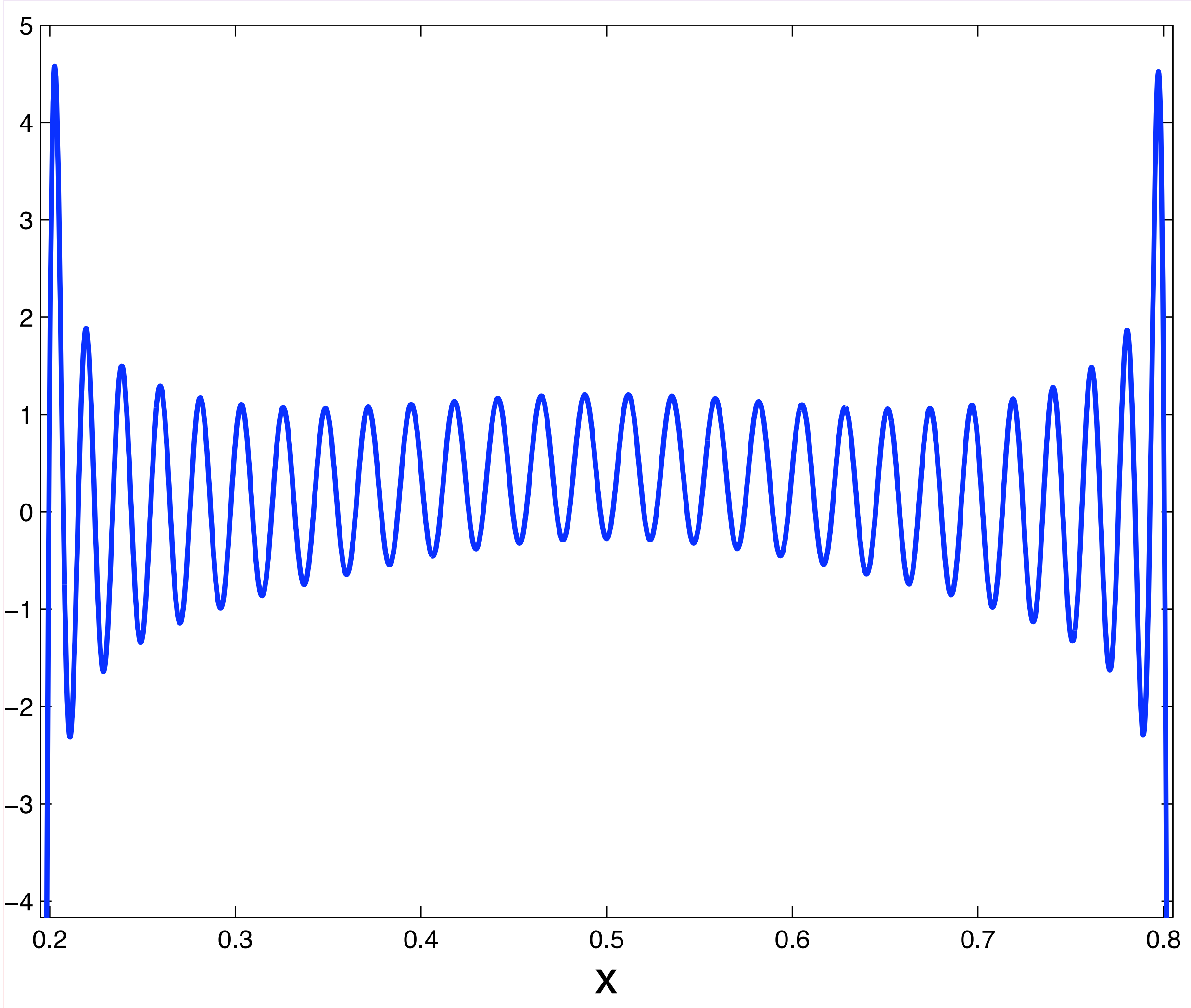
$$v = \hat{\varphi}$$

where $\hat{\varphi}$ is the solution of the adjoint system corresponding to the minimizer is the control such that

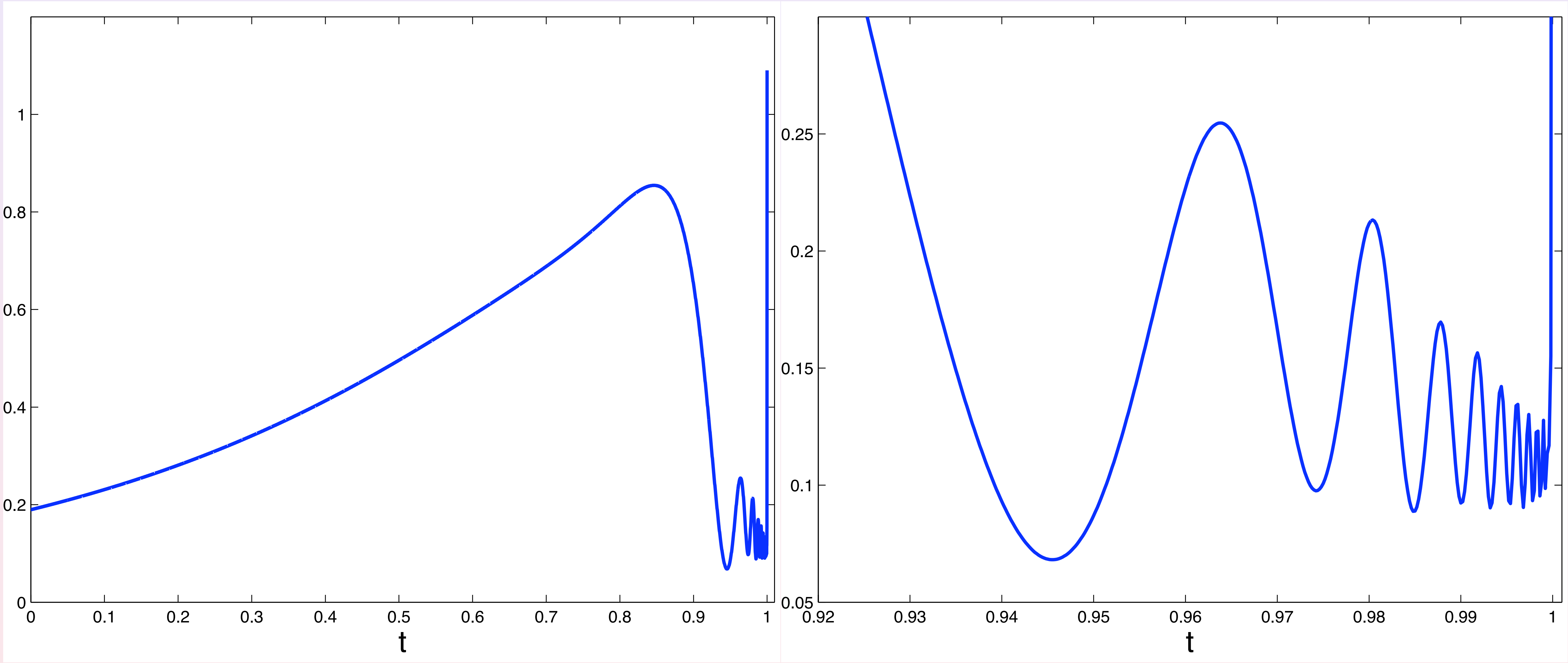
$$y(T) \equiv 0.$$

As a consequence of this, even if the control is in L^2, the data φ^0 of the adjoint system at time T (which is surely in $\mathcal{H}$) tend not to be in any reasonable space, thus making computations very hard.
$T = 1$, $\omega = (0.2, 0.8)$: $\varphi^{0,M}$ for $M = 80$ on Ω (**Left**) and on ω (**Right**).





$T = 1$, $\omega = (0.2, 0.8)$: $\|\varphi^M(\cdot, x)\mathcal{X}_\omega(x)\|_{L^2(\Omega)}$ for $M = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).



Several remedies have been derived in the literature, starting with the pioneering work by R. Glowinski and J. L. Lions. One of them is based on Tychonnoff regularization. It consists on adding a regularizing term to the functional to be minimized (or its discrete version):

$$J_0^{\varepsilon}(\varphi^0) = \frac{1}{2}\int_0^T\int_{\omega}\varphi^2 dxdt + \frac{\varepsilon}{2}||\varphi^0||_{L^2}^2 + \int_{\Omega}\varphi(0)u^0 dx. \quad (9)$$

One can prove that, whenever the minimizer of the original functional J belongs to L^2, then the regularized controls converge polynomially as ε tends to zero.
But, as the numerical experiments show, the minimizer does not belong to L^2 not even to any H^{-s}. [2]

[2] S. Micu & E. Z. Regularity issues for the null-controllability of the linear 1-d heat equation, Systems and Control Letters, 2011.

Kannai transform allows transfering the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-s^2/4t} W(s)ds$$

where $W(x, s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

$$W_{ss} - \Delta W = 0 \quad + \quad K_t - K_{ss} = 0 \quad \rightarrow \quad U_t - \Delta U = 0,$$

$$W_{ss} - \Delta W = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \rightarrow \quad iU_t - \Delta U = 0.$$

This can be actually applied in a more general abstract context $(U_t + AU = 0)$ but not when the equation has time-dependent coefficients.

This can also be used in the context of control:

[Control of the wave equation in Ω]

$$+$$

[$1-d$ controlled fundamental solution of the heat equation]

$$\Longrightarrow$$

[Control of the heat equation in Ω].

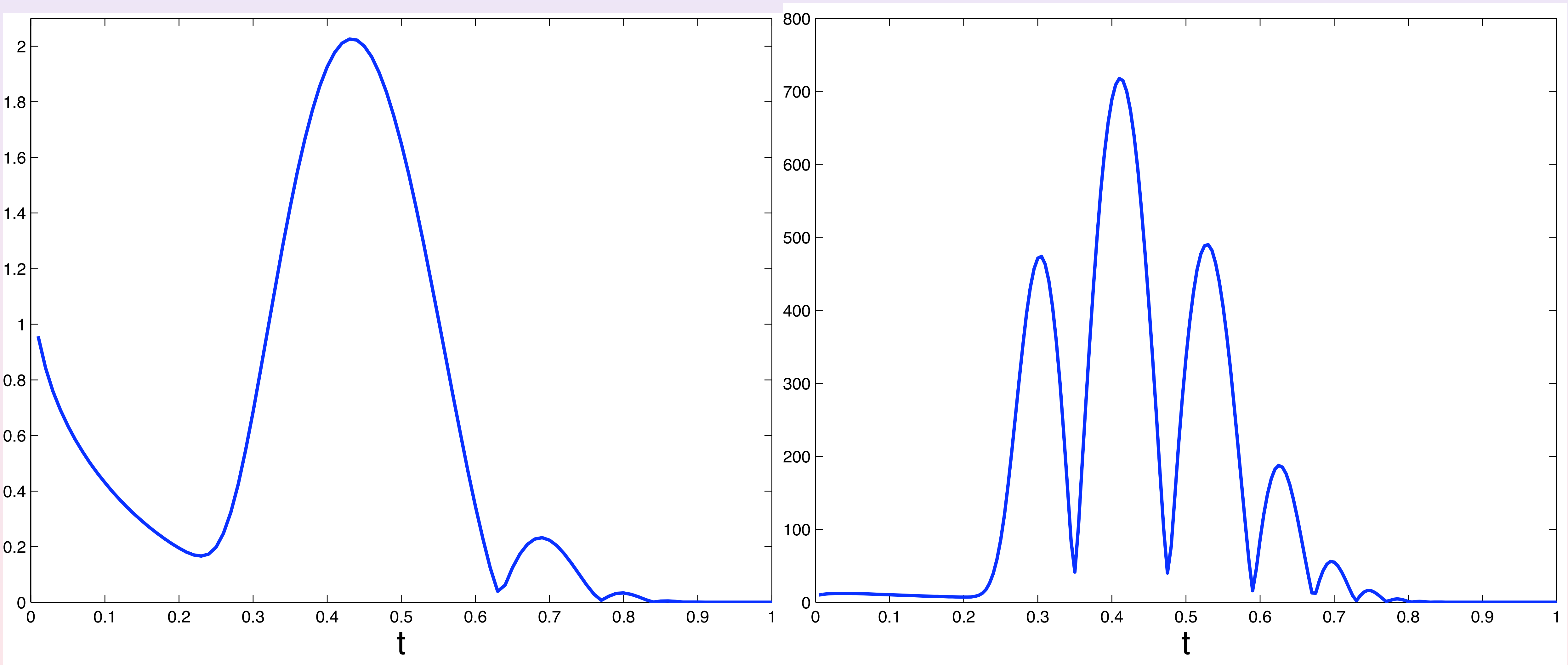
In a recent paper in collaboration with A. Münch we propose a different strategy based on the following facts:[3]

- A lot of work has been done to build efficient algorithms to compute exact controls for the wave equation.
- The Kannai transform allows to construct the control of the heat equation by convolution of the wave one with a $1-d$ heat kernel.

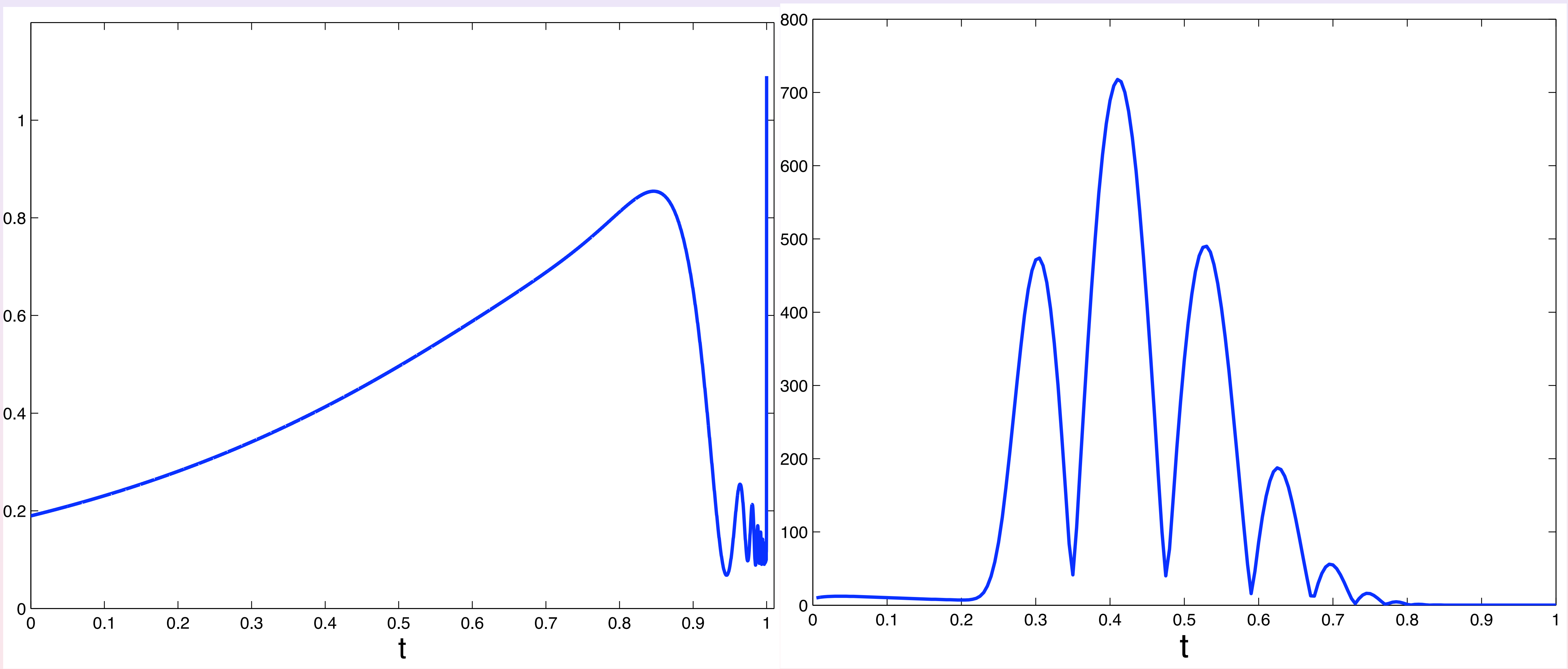
The method is laborious to be developed numerically but turns out to be efficient.

[3] A. Münch and E. Z. Numerical approximation of null controls for the heat equation through transmutation, J. Inverse Problems, 2010.

$L^2(\omega)$-norm of the control v vs time t for $(y_0(x), T, c) = (\sin(\pi x), 1, 1/10)$ (**Left**) and $(y_0(x), T, c) = (\sin(3\pi x), 1, 1/5)$ (**Right**).



Standard L^2-control vs Kannai control.



Once more the wining strategy is a smart combination of the continuous and discrete approaches.

Conclusions

- Efficient and rigorous numerical computation of controllers can be built but often combining tools from the continuous and the discrete approaches.
- Plenty is still to be done in the interfaces between PDE, Control, Numerics, Harmonic Analysis,...

Perspectives

- Multi-resolution filtering techniques.
- Numerical control of waves in random media and in the presence of noise.
- Robust controllers.
- Discrete version of Geometric Optics?
- Efficient solvers of the ill-posed heat equation
- Multiphysics systems: thermoelasticity, fluid-structure interaction,...
- Multiscale control (micro/macro),...