CONTROL OF INFINITE-DIMENSIONAL SYSTEMS (E. Zuazua, 2021)

Recall first the finite-dimensional control system:

$$x' + Ax = Bu, \quad 0 < t < T, \quad A \in M_{n \times n}, \; B \in M_{n \times m}, \quad m < n$$

We know that the system is controllable if and only if it fulfills the rank condition

$$\operatorname{rank} \left[B, AB, \ldots, A^{n-1} B \right] = n.$$

What should be the extension to infinite-dimensional systems?

A FIRST EXAMPLE IN ℓ^2

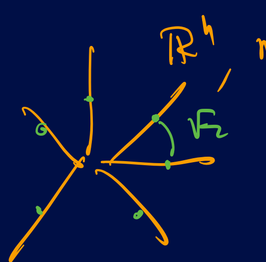
Let us first consider the prototypical example in which, instead of working in $\mathbb{R}^n$, we do it in the Hilbert space ℓ^2:

$$\ell^2 = \left\{ \{a_j\}_{j \geq 1} \; : \; \sum_{j \geq 1} a_j^2 < \infty \right\}$$

$$\left(\{a_j\}, \{b_j\} \right)_{\ell^2} = \sum_{j=1}^{\infty} a_j b_j$$

$$\leq \left(\sum_{j \geq 1} a_j^2 \right)^{1/2} \left(\sum_{j \geq 1} b_j^2 \right)^{1/2}$$

ℓ^2



The canonical basis is given by $\{\vec{e}_\ell\}_{\ell \geq 1}$ where

$\vec{e}_\ell = \{0, 0, \ldots 0, 1, 0 \ldots 0 \ldots\}$ ↳ ℓ-th component

$\vec{e}_j \cdot \vec{e}_\ell = 0 \quad j \neq \ell$

$\{\vec{e}_j\}_{j \geq 1} \subset \ell^2 : \|\vec{e}_j\|_{\ell^2} = 1$

$$\|\vec{e}_j - \vec{e}_\ell\|_{\ell^2} = \sqrt{2}, \; j \neq \ell$$

Consequently $\{\vec{e}_j\}_{j \geq 1}$ does not have any Cauchy or convergent subsequence.

However $\{\vec{e}_j\}$ tends to zero in the weak topology of ℓ^2.

But the convergence cannot hold in the strong topology

$$\|e_j\|_{\ell^2} = 1, \; \forall j \geq 1; \; e_j \to 0, \text{ weakly in } \ell^2, \; j \to \infty.$$

Consider now an infinite-dimensional system in ℓ^2.

$$\begin{cases} x' + Ax = Bu \\ x(0) = x_0 \end{cases}$$

$x_0 \in \ell^2$

$x(t) \in \ell^2$: $\quad A : \ell^2 \to \ell^2, \; B : \mathbb{R}^m \to \ell^2$

$$A = (a_{ij})_{i \geq 1, j \geq 1}, \; B \in (b_{ij})_{\substack{i \geq 1 \\ 1 \leq j \leq m}}$$

The goal remains the same: Given $T > 0$ and $x_T \in \ell^2$ find $u \in L^2(0,T;\mathbb{R}^m)$ such that $x(T) = x_T$.

Assume that

$$\|Ax\|_{\ell^2} \leq \|A\| \, \|x\|_{\ell^2} \; ; \; \|Bu\| \leq \|B\| \, \|u\|.$$

Then, the variation of constants formula applies

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} B u(s)\, ds$$

with

$$e^{At} = \sum_{j \geq 0} \frac{t^j}{j!} A^j,$$

and

$$\|e^{tA}\| \leq e^{|t| \, \|A\|}.$$

The condition

$$\text{rank}\,[B, AB \dots A^{n-1}B \dots] = \infty$$

would be natural since Cayley-Hamilton's Theorem does not apply. It would guarantee that an infinite number of components of the system are controllable. But it would not suffice to show that the full system is controllable. Finite or infinitely many other components could fail to be controllable in a situation similar to:

$$\begin{cases} x_1' + x_1 = 0 \\ x_2' + 2x_2 = u \end{cases}$$

THE TRANSPORT EQUATION

$$f_t + V \cdot \nabla_x f = 0$$

The results we have on the control of finite-dimensional systems can be interpreted in terms of the control of the transport equation, but the control being on the velocity field V.

Indeed, the n-dimensional system

$$x' + Ax = Bu$$

is the one of the characteristics of the n-dimensional transport equation

$$f_t + (-Ax + Bu)\cdot \nabla_x f = 0.$$

The map $u \to f$, which has a bilinear structure, is nonlinear. So, in terms of the transport equation this constitutes a nonlinear control problem.

Observe also that the control of the transport equation from an initial configuration $f_0(x)$ to a final one $f_T(x)$, would require the control of all the characteristic lines, simultaneously, with the same control $u(t)$. And this is impossible. Therefore the class of velocity fields $V(x,t)$ needed to control the transport equation would be much larger.

THE WAVE EQUATION

Consider now the wave equation

$$\begin{cases} y_{tt} - \Delta y = 0 & \Omega \times (0,T) \\ y|_{\partial\Omega} = 0 & \\ y(0) = y_0,\ y_t(0) = y_1 & \text{in } \Omega \end{cases}$$

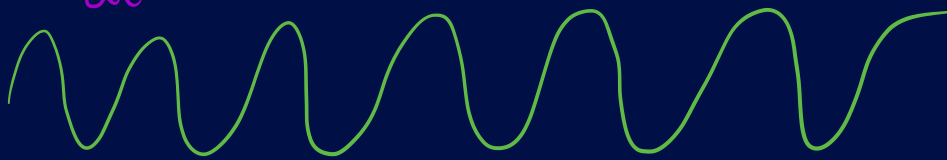
$\Omega \subset \mathbb{R}^d$

The energy

$$E(t) = \frac{1}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{2}\int_\Omega |y_t|^2 dx$$

is conserved along trajectories

$$\frac{dE}{dt}=0 \iff E(t)=E_0.$$



The wave equation can be interpreted as an infinite-dimensional harmonic oscillator

Consider now the damped wave equation

$$\begin{cases} y_{tt}-\Delta y + y_t = 0 & \Omega\times(0,\infty) \\ y|_{\partial\Omega}=0 & \\ y(0)=y_0,\quad y_t(0)=y_1 & \Omega \end{cases}$$

It can be rewritten as

$$y_{tt}-\Delta y = -y_t = f$$ → Applied force = velocity damping

which is the infinite-dimensional analog of

$$x''+x+x'=0.$$

The total energy

$$E(t)=\overbrace{\frac{1}{2}\int_\Omega |\nabla y|^2\,dx}^{\text{potential energy}}+\overbrace{\frac{1}{2}\int_\Omega |y_t|^2\,dx}^{\text{kinetic energy}}$$

is constituted by the superposition of the potential and kinetic energy.

The energy dissipation law reads:

$$\frac{dE}{dt}(t)=-\int_\Omega |y_t|^2\,dx \le 0$$

which decreases as time t increases.

But, does this damping mechanism produce an effective decay?

- $E(t) \to 0, \quad t \to \infty$?
- $E(t) \leq C e^{-\alpha t} E_0, \quad t \to \infty$

Obviously, if we had

$$\frac{dE}{dt} \leq -\alpha E$$

we would also have the exponential decay

$$E(t) \leq \exp(-\alpha t) E_0.$$

But this is far from obvious since the damping term, apparently, does not dissipate the potential energy.

$$\left[\frac{dE}{dt} \leq -\alpha E \Leftrightarrow \frac{dE}{dt} + \alpha E \leq 0 \overset{e^{\alpha t}}{\Leftrightarrow} \frac{d}{dt}\left(e^{\alpha t} E(t)\right) \leq 0 \right.$$

$$\left. \to e^{\alpha t} E(t) \leq E(0) \Rightarrow E(t) \leq e^{-\alpha t} E_0 \right]$$

The danger to avoid would be that the damped wave equation inherits the structure of a partially damped harmonic oscillator system:

$$\begin{cases} x'' + x = 0 & \longrightarrow \text{undamped conservative component} \\ y'' + 2y + y' = 0. & \longrightarrow \text{damped exponentially decaying component.} \end{cases}$$

The question we address is the following:

Does the velocity damping on the wave equation have an impact on the potential energy as well, sufficient to efficiently dissipate the whole energy?

The Lyapunov functional approach.

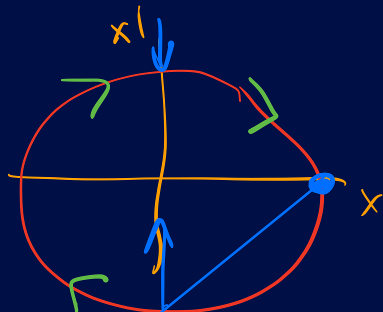
Goal: Find a perturbation of the energy $E_\varepsilon \sim E$ s.t.

$$\frac{dE_\varepsilon}{dt} \le -\alpha E$$

So to make it visible that the velocity damping has also an impact on the potential energy.

This is successfully achieved for the harmonic oscillator by means of the velocity damping

$$x'' + x + x' = 0.$$



And, as we have seen, this is just one example of a much more general setting where in

$$x' + Ax = Bu$$

A is s.t. $A^* = -A$ and $(A,B) \to$ Kalman cond.

$$\begin{cases} x' + Ax = -BB^*x \\ A = -A^* \end{cases}$$

The Lyapunov function technique we develop takes advantage (at the PDE level, in the context of the wave equation), of the same interaction between A and B that in the ODE setting leads to the exponential decay.

Consider the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\, E(t)$$

with $\varepsilon > 0$ small so that:

$$\begin{cases} E_\varepsilon \sim E \\ \frac{1}{2} E \leq E_\varepsilon \leq \frac{3}{2} E \end{cases}$$

This can be easily achieved by means of Cauchy-Schwarz and Poincaré inequalities:

$$\left|\varepsilon \int_\Omega y y_t \, dx\right| \leq |\varepsilon| \int_\Omega |y||y_t| \, dx \leq \frac{|\varepsilon|}{2}\left[\int_\Omega |y_t|^2 + \int_\Omega |y|^2 dx\right]$$

$$\boxed{\begin{array}{c}\text{Poincaré} \\ \text{in } H^1_0(\Omega) \\ \Omega \text{ being bounded}\end{array}} \rightarrow \leq \frac{\varepsilon}{2}\left[\int_\Omega |y_t|^2 dx + C(\Omega)\int_\Omega |\nabla y|^2 dx\right]$$

$$\leq \varepsilon C\, E(t).$$

This shows that $E \sim E_\varepsilon$ for $\varepsilon > 0$ small enough. In this manner the exponential decay of $E(t)$ is reduced to that of $E_\varepsilon(t)$. As we shall see, the added term $\int_\Omega y y_t dx$ allows to make it explicit the dissipativity effect that the velocity damping has on the potential energy, something that does not emerge in the energy dissipation law.

We have

$$\frac{dE_\varepsilon}{dt} = \frac{dE}{dt} + \varepsilon \frac{d}{dt}\int_\Omega y y_t \, dx$$

$$= -\int_\Omega |y_t|^2 dx + \varepsilon \int_\Omega y_t^2 dx + \varepsilon \int_\Omega y\, y_{tt}\, dx$$

$$= -\int_\Omega |y_t|^2 dx + \varepsilon \int_\Omega |y_t|^2 dx + \varepsilon \int_\Omega y\,(\Delta y - y_t)\, dx$$

$$= -(1-\varepsilon)\int_\Omega |y_t|^2 dx - \varepsilon \int_\Omega |\nabla y|^2 dx - \varepsilon \int_\Omega y\, y_t$$

$$\frac{dE_\varepsilon}{dt} = \underbrace{-(1-\varepsilon)\int_\Omega |y_t|^2 - \varepsilon \int_\Omega |\nabla y|^2 dx}_{\leq -2\min(\varepsilon,\, 1-\varepsilon) E(t)} - \varepsilon \int_\Omega y\, y_t\, dx$$

$$\left| \varepsilon \int_\Omega y\, y_t\, dx \right| \leq \frac{1}{2}\int_\Omega |y_t|^2 + \frac{\varepsilon^2}{2}\int_\Omega |y|^2 dx$$

$$\leq \frac{1}{2}\int_\Omega |y_t|^2 dx + \frac{C\varepsilon^2}{2}\int_\Omega |\nabla y|^2 dx$$

$$\frac{dE_\varepsilon}{dt} \leq -\left(1 - \varepsilon - \frac{1}{2}\right)\int_\Omega |y_t|^2 dx - \left(\varepsilon - \frac{C\varepsilon^2}{2}\right)\int_\Omega |\nabla y|^2$$

$$\overset{0<\varepsilon\ll 1}{\leq} -\frac{1}{4}\int_\Omega |y_t|^2 - \frac{\varepsilon}{2}\int |\nabla y|^2 dx$$

$$\leq -\varepsilon E(t) \leq -\frac{\varepsilon}{2} E_\varepsilon(t)$$

In this way we conclude the exponential decay of E_ε, which also implies the exponential decay of $E(t)$.

This idea of augmenting the energy to make more visible the dissipative effects on the various components of the system can be extended to many other models.

The following is a natural interpretation in the context of finite-dimensional systems (A, B) fulfilling the Kalman rank condition:

$$(A, B) \text{ controllable} \iff \text{rank}\,[B, AB, \dots A^{n-1}B] = n$$

$$\iff \ker\,[B^*, B^*A^* \dots (A^*)^{n-1}B] = 0.$$

This means that the euclidean norm $\|\cdot\|_{\mathbb{R}^n}$ is equivalent to the "Kalman or rank norm":

$$\|x\|_K^2 = \sum_{j=0}^{n-1} \varepsilon_j \|B^*(A^*)^j x\|_{\mathbb{R}^n}^2$$

for any choice of the constants $\varepsilon_j > 0$, $j = 0, \dots, n-1$.

The method based on comparing E to E_ε is therefore also related to other classical PDE concepts:

- hypocoercivity (Desvillettes – Villani)
- hypoellipticity (Hörmander)

PDEs are classically classified as being elliptic (Δ_x = Laplacian), parabolic ($\partial_t - \Delta_x$ = heat) and hyperbolic ($\partial^2_t - \Delta_x$ = wave).

But the class of hypoelliptic PDEs is worth underlining too. It is not fully parabolic but it ensures that the fundamental solution is C^∞-smooth.

The most classical and paradigmatic example of hypoelliptic PDE is the Kolmogorov equation:

$$f_t - f_{xx} - x f_y = 0 \quad \text{(Kolmogorov).}$$

We observe that it is a mix of

$$f_t - f_{xx} = 0 \rightarrow \text{Heat equation = parabolic}$$

$$f_t - x f_y = 0 \rightarrow \text{Transport equation = hyperbolic.}$$

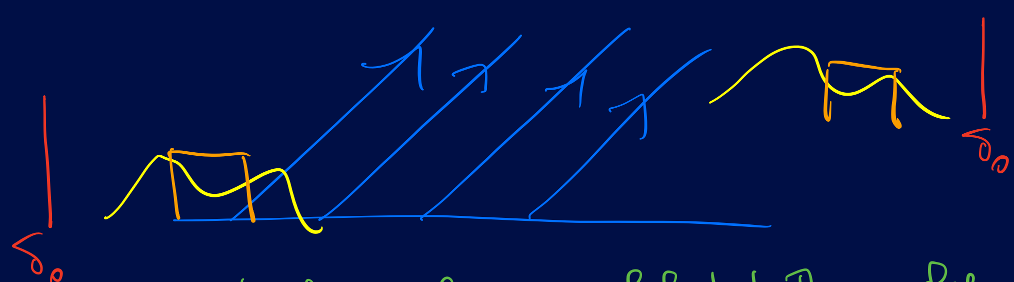
The regularizing effect of parabolic equations is well represented by the Gaussian fundamental solution of the heat equation:

Regularizing effect from the singular measure of the Dirac delta δ_0 at $t=0$ to the analytic gaussian profile at $t>0$.

$$G(x,t) = (4\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

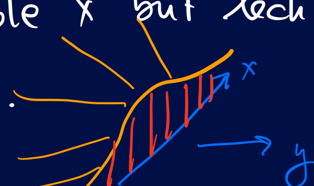
However, the transport equation lacks of any regularizing effect:

$$f_t + v f_x = 0 \to f(x,t) = f_0(x - vt).$$



Absolute lack of regularizing effect! The profile of the initial datum is preserved along the evolution, independent of how singular it is.

For the Kolmogorov equation we could expect a combined effect: Regularization in the space variable x but lack of any smoothing effect on the variable y.



But Kolmogorov discovered the hypoelliptic effect computing explicitly the fundamental solution, that turns out to be a Gaussian both in x and in y.

Why? Our first intuition of the parabolic-hyperbolic effect without mixing would be correct for a model of the form

$$f_t - f_{xx} - f_y = 0$$

because the two involved operators $-\partial_x^2$ and ∂_y commute. When solving

$$f_t + A_1 f + A_2 f = 0$$

we have

$$f(t) = e^{(A_1+A_2)t} f_0$$

and when A_1 and A_2 commute ($[A_1, A_2] = A_1A_2 - A_2A_1 = 0$) we have

$$e^{(A_1+A_2)t} = e^{A_1 t} e^{A_2 t} = e^{A_2 t} e^{A_1 t}.$$

But the two operators involved in the Kolmogorov model, namely $-\partial_x^2$ and $-x\partial_y$, do not commute. Actually

$$[-\partial_x^2, -x\partial_y] = [\partial_x^2, x\partial_y] = \partial^2_{xy}.$$

The lack of commutation generates a diffusion effect in the oblique direction $x+y$ that, together with the predominant presence of the diffusion ∂_x^2 in the variable x, generates a diffusion effect in the whole plane $(x,y) \in \mathbb{R}^2$.

Note however that the diffusion generated by the lack of commutation is a second order effect and, therefore, milder.

SPLITTING AND DOMAIN DECOMPOSITION

The lack of commutativity of the various operators entering in an evolution equation has led to various computational methods.

Consider the Cauchy problem

$$\begin{cases} x' + A_1 x + A_2 x = 0 \quad , t > 0 \\ x(0) = x_0 \end{cases}$$

whose solution is

$$x(t) = e^{(A_1 + A_2)t} x_0 .$$

This expression has fully sense when A_1 and A_2 are bounded operators and for most relevant PDE systems can be justified in the context of semigroups.

When A_1 and A_2 commute (i.e. $[A_1, A_2] = 0$) the solution can be decomposed as

$$x(t) = e^{(A_1+A_2)t} x_0 = e^{A_1 t} e^{A_2 t} x_0 = e^{A_2 t} e^{A_1 t} x_0 .$$

This means that the solution of the combined system can be computed as the output of the consecutive resolution of both independent systems (in any order): $x(T) = e^{A_1 T} e^{A_2 T} x_0$

$$x' + A_1 x = 0 , \quad 0 < t < T, \quad x(0) = x_0 \quad \rightarrow x(T) = e^{A_1 T} x_0$$

$$x' + A_2 x = 0 , \quad 0 < t < T ; \quad x(0) = e^{A_1 T} x_0 \rightarrow x(T) = e^{A_2 T} e^{A_1 T} x_0 .$$

When A_1 and A_2 do not commute, the effect of the non-trivial commutator $[A_1, A_2]$ can be observed in the second order term of the power series expansion of the exponential:

$$e^{A_1 + A_2} = I + (A_1 + A_2) + \frac{1}{2}(A_1 + A_2)^2 + \dots = I + A_1 + A_2 + \frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 + A_1 A_2 + A_1 A_2 + \dots$$

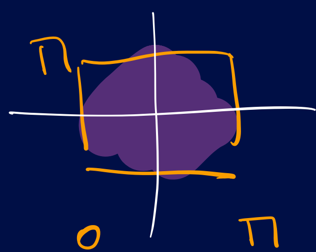
But

$$e^{A_1} e^{A_1} = \left(I + A_1 + \frac{A_1^2}{2} \dots \right) \left(I + A_2 + \frac{A_2^2}{2} \dots \right)$$

$$= I + (A_1 + A_2) + \frac{A_1^2}{2} + \frac{A_2^2}{2} + A_1 A_2$$

The same occurs when solving a PDE in a complex geometry.

$-\Delta$



$\sin(kx)\sin(my)$

$\lambda = k^2 + m^2$

→ Solution of the Dirichlet problem for the Laplacian is separated variables in a square domain

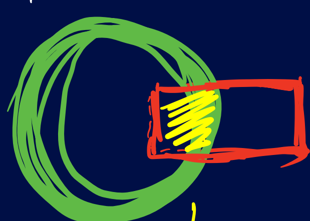
$\Omega = [0, \pi]^2$

$\varphi_k(r)\, e^{il\theta}$

→ Solution of the Laplacian in a disk in spherical coordinates

Domain Decomposition (DD) can be used in more complex domain where there is no a natural coordinate system

$-\Delta\varphi = f \quad \Omega$



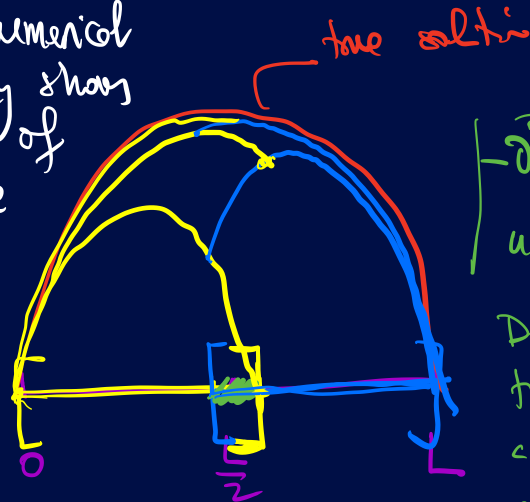
How to solve the Laplace equation in a domain constituted by the overlapping of the disk and the rectangle?

↳ overlapping region

↳ disk

→ rectangle

An explicit or numerical computation easily shows the convergence of a splitting iterative method.



$$\boxed{\begin{aligned} -\partial_x^2 u &= 1 \quad (0,L) \\ u(0) = u(L) &= 0 \end{aligned}}$$

Dividing the segment in two sub-segments with a small overlap interval and solving each problem in an iterative manner leads rapidly to the solution.

Lie splitting leads to the same results for time-dependent problems

$$e^{A_1+A_2} = \lim_{n\to\infty} \left(e^{\frac{A_1}{n}} e^{\frac{A_2}{n}} \cdots e^{\frac{A_1}{n}} e^{\frac{A_2}{n}} \right)$$

LOCALIZED DAMPING: EITHER ON THE BOUNDARY OR IN THE INTERIOR

The wave equation with damping everywhere in domain is not the most interesting example in applications where one aims at feedback mechanisms localized on the support of actuators:

Ω

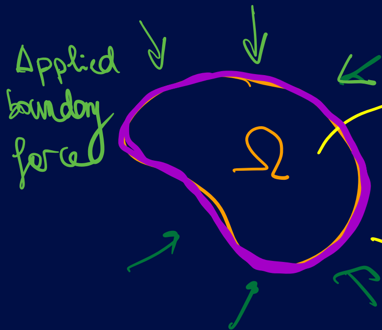
$$y_{tt} - \Delta y + y_t = 0$$

$$\to y_{tt} - \Delta y = -y_t$$

Actuation everywhere in the domain Ω, i.e. at each $x \in \Omega$ we have an actuator able to realize

$$f(x,t) = -y_t(x,t).$$

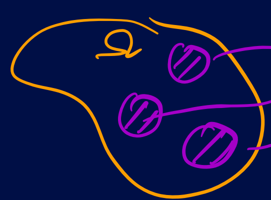
→ In many applications (like acoustic barriers) the damping is localized on the boundary.



→ Elastic body or medium in which acoustic waves propagate.

→ damping localized on the boundary.

→ Boundary damping



→ actuators damping or controlling waves

→ Damping localized on interior actuators.

Given the domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and a subdomain $\omega \subset \Omega$, consider the wave equation with damping localized in ω:

$$\begin{cases} y_{tt} - \Delta y + 1_\omega y_t = 0 & \Omega \times (0,\infty) \\ y|_{\partial\Omega} = 0 & \\ y(0) = y_0, \; y_t(0) = y_1 & \Omega \end{cases}$$

1_ω = Characteristic function of ω

$$= \begin{cases} 1 & \text{in } \omega \\ 0 & \text{outside} \end{cases}$$

$$y_{tt} - \Delta y = f(x,t) = \begin{cases} 0 & \text{in } \Omega \setminus \omega \\ -y_t & \text{in } \omega \end{cases}$$

Propagating waves are damped only when the signal reaches ω.

Question: Can a localized damping mechanism be efficient:

- $E(t) \to 0, \quad t \to \infty$
- $E(t) \leq C e^{-\alpha t} E_0, \quad \forall t > 0.$

[If a dissipative semigroup decays, then the decay needs to be exponential.

$$S(t) : X \to X$$

$$\|S(t)\|_{\mathcal{L}(X,X)} \leq 1, \quad \forall t > 0.$$

Wave equation

$X = H_0^1(\Omega) \times L^2(\Omega)$,

$y_{tt} - \Delta y + y_t = 0$

$$\begin{cases} y_t = z \\ z_t = \Delta y - z \end{cases}$$

$x'' + x + x' = 0$

$x_0 \in X \to x(t) = S(t) x_0.$

$$\|S(t_2) x_0\|_X \leq \|S(t_1) x_0\|_X, \quad \text{if } t_2 > t_1.$$

If $t_2 = t_1 + \tau, \ \tau > 0,$

$$S(t_2) x_0 = S(t_1 + \tau) x_0 = S(\tau)\left[S(t_1) x_0\right]$$

$$\|S(t_2) x_0\|_X = \|S(\tau)(S(t_1) x_0)\|_X \leq \|S(\tau)\|_{\mathcal{L}(X,X)} \|S(t_1) x_0\|_X$$

$$\leq \|S(t_1) x_0\|_X$$

Assume that the semigroup decays

$$\|S(t)\|_{\mathcal{L}(X,X)} = \phi(t) \to 0, \quad t \to \infty.$$

$\exists\, T > 0$:

$$|\phi(T)| = \gamma < 1, \quad (T \gg 1)$$

$$\|S(kT)\|_{\mathcal{L}(X,X)} = \|S(T)(S((k-1)T)\|_{\mathcal{L}(X,X)} \cdots$$

$$\leq \gamma^k.$$

$t > 0$; $t \gg 1$, $\quad t = kT + \tau, \quad 0 < \tau < T$

$$\|S(t)\|_{\mathcal{L}(X,X)} = \|S(kT+\tau)\|_{\mathcal{L}(X,X)}$$

$$\leq \|S(kT)\|_{\mathcal{L}(X,X)} \leq \gamma^k = e^{\log(\gamma) k}$$

$$= e^{-|\log(\gamma)| k} = e^{-|\log(\gamma)| \left(\frac{t-T}{T}\right)}$$

$$= C_T e^{-\frac{|\log(\gamma)|}{T} t}$$

Then the exponential decay rate is $\sim \dfrac{|\log(\gamma)|}{T}$.

For instance, for the wave equation with dissipation, if

$$E(t) \leq \phi(t) E_0.$$

with $\quad \phi(t) \to 0$ as $t \to \infty$ then necessarily

$$E(t) \leq C e^{-\alpha t} E_0.$$

What about semigroups with slow decay?

$\to$ spectral representation of a dissipative semigroup in ℓ^2 for which the damping effect is weak at high frequencies

λ_k: $\quad \mathrm{Re}(\lambda_k) < 0 \quad ; \quad \mathrm{Re}(\lambda_k) \to 0, \quad k \to \infty.$

- $E(t) \to 0, \; t \to \infty, \quad \forall$ solution
- But $\; E(t) \leq C e^{-\alpha t} E_0 \to$ impossible!

Both results are compatible since the fact that

$$E(t) \to 0 \quad \text{for all solutions}$$

is much weaker than

$$\| S(t) \|_{\mathcal{L}(X,X)} = \phi(t) \to 0.$$

When

$$E(t) \to 0, \quad \text{as } t \to \infty,$$

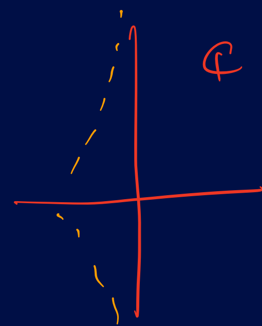
the decay rate may depend on the solution and be compatible with the fact that

$$\|S(t)\|_{\mathcal{L}(X,X)} = 1, \quad \forall t>0.$$

ℓ^2: $\{\lambda_k\}$ as above

$$x_0 = \sum_{k\geq 1} x_{0,k}\,\vec{e}_k$$

$$x(t) = \sum_{k\geq 1} x_{0,k}\, e^{\lambda_k t}\,\vec{e}_k$$



$$\|x_0 \to x(t)\|_{\mathcal{L}(\ell^2,\ell^2)} = 1.$$

$$\|x(t)\|^2_{\ell^2} = \sum_{k\geq 1} |x_{0,k}|^2 e^{2\lambda_k t} \underset{\substack{\uparrow \\ \mathrm{Re}\,\lambda_k<0}}{\leq} \sum_{k\geq 1} |x_{0,k}|^2 = \|x_0\|^2_{\ell^2}$$

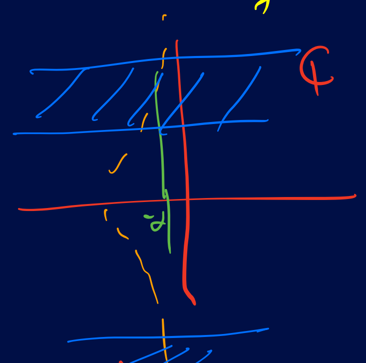
$$\to \|S(t)\|_{\mathcal{L}(\ell^2,\ell^2)} \leq 1.$$

On the other hand

$$\|S(t)\|_{\mathcal{L}(\ell^2,\ell^2)} = \gamma < 1$$

is impossible. Thus $\|S(t)\|_{\mathcal{L}(\ell^2,\ell^2)} = 1.$

Warning: For infinite-dimensional systems, the decay of each trajectory is compatible with the absence of decay of the semigroup.



Ex. 1.

Ex 2.

$\mathrm{Re}\,\lambda_k \to 0$, $k \to \infty$	$\mathrm{Re}\,\lambda_k \le -\alpha < 0$, $\forall k$.
$\Downarrow$	$\Downarrow$
Arbitrarily slow decay rate.	Exponential decay rate
Asymptotic Stability	Exponential stability.

Exponential stability is much more useful from a practical perspective since it guarantees that all frequencies are effectively damped exponentially, uniformly.

Remark: Note that if, for instance, for a damped wave equation we had

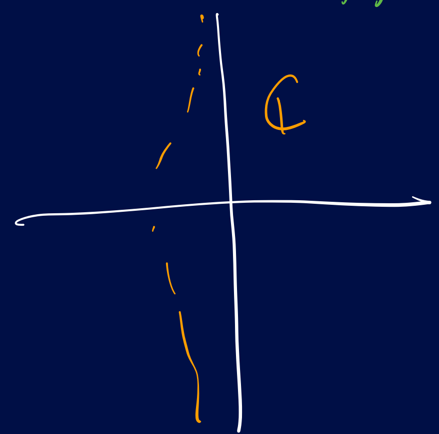
$$E(t) \le \frac{1}{1+t} E_0, \quad \forall t > 0, \ \forall \text{ solution}$$

$$\sim t^{-1}$$

then, there would exist $C>0$ and $\alpha>0$ s.t.

$$E(t) \leqslant C e^{-\alpha t} E_0.$$

There are however situations where the decay might be slow. In that case the norm of the semigroup decays, but not as a bounded operator from X to X but from the domain of the generator $D(A)$ onto X: $\|S(t)\|_{\mathcal{L}(D(A),X)} \leqslant \frac{1}{1+t}$.



$\mathrm{Re}(\lambda_k) < 0$

$\lim_{k \to \infty} \mathrm{Re}(\lambda_k) = 0.$

$$S(t) x_0 = \sum_{k \geqslant 1} x_{0,k}\, e^{\lambda_k t} \vec{e}_k.$$

- $\|S(t)\|_{\mathcal{L}(\ell^2, \ell^2)} = 1$
- There is no uniform exponential decay.
- But we can achieve some decay if we impose further conditions on x_0.

$$\operatorname{Re} \lambda_k = -\frac{1}{k}.$$

$$x_0 \in \ell^2 \iff \sum_{k\geq 1} |x_{0,k}|^2 < \infty$$

$$\in h^1 \iff \sum_{k \geq 1} k^2 |x_{0,k}|^2 < \infty.$$

The latter stronger summability condition allow to achieve some decay rate:

$$\Big\| x(t) = \sum_{k\geq 1} x_{0,k}\, e^{\lambda_k t} \vec{e}_{0,k} \Big\|_{\ell^2}^2 = \sum_{k\geq 1} k^2 |x_{0,k}|^2 \left(\frac{e^{-\frac{2t}{k}}}{k^2} \right)$$

$$\leq \underbrace{\left(\sum k^2 |x_{0,k}|^2 \right)}_{\|x_0\|_{h^1}^2} \sup_{k \geq 1} \left(\frac{e^{-\frac{2t}{k}}}{k^2} \right)$$

$$\frac{e^{-\frac{2t}{k}}}{k^2} = \frac{e^{-\frac{2t}{k}}}{t^2 \left(\frac{k}{t}\right)^2} \underset{\frac{t}{k} = s}{=} \frac{1}{t^2} \underbrace{\left(\frac{e^{-2s}}{s^2} \right)}_{\leq C t^{-2}.}$$

Thus, for such a spectrum ($\operatorname{Re} \lambda_k \sim -k^{-1}$) we have

$$\|x(t)\|_{\ell^2} \leq \frac{C}{t} \|x_0\|_{h^1}$$

Note that this does not provided information on the decay of $S(t) : \ell^2 \to \ell^2$. In fact $\|S(t)\|_{\mathcal{L}(\ell^2, \ell^2)} = 1 \ \forall t > 0$.

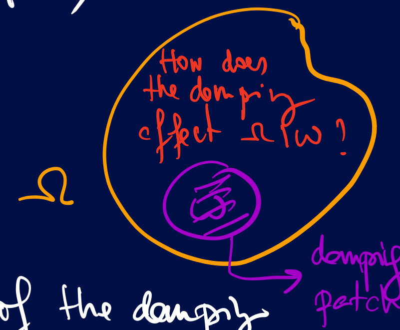
$$\begin{cases} y_{tt} - \Delta y + 1_\omega y_t = 0 \\ y|_{\partial\Omega} = 0 \\ y(0) = y_0, \quad y_t(0) = y_1 \end{cases}$$

The works of C. Dafermos, A. Haraux ... in the 70's showed that the energy of each solution of the wave equation, $E(t)$, tends to zero as $t \to \infty$ provided the damping is effective on a measurable set ω with $|\omega| > 0$.

$$E(t) = \frac{1}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{2}\int_\Omega |y_t|^2 dx$$

In this case we have

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx$$



but it is challenging to quantify the impact of the damping away from ω.

When $\omega = \Omega$, $E \mapsto E_\varepsilon$ (hypoellipticity), and it's easy to observe the dissipative effect on the potential energy $\to$ exponential decay.

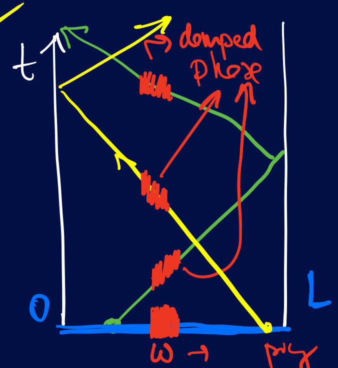
Can we, out of the dissipation of the energy on ω, realize the effective dissipation of the kinetic energy everywhere in Ω?

In other words

Does a localized damping produce a global decay?

This problem is of infinite-dimensional nature and cannot be handled through rank conditions.

For the 1-d D'Alembert equation modelling the vibrations of a flexible string the interpretation of solutions as the superposition of two travelling waves ($f(x+t)+g(x-t)$) allows to see that a localized damping term may effectively damp the energy



$$\boxed{\begin{aligned} &y(x,t) \\ &= f(x+t) + g(x-t) \end{aligned}}$$

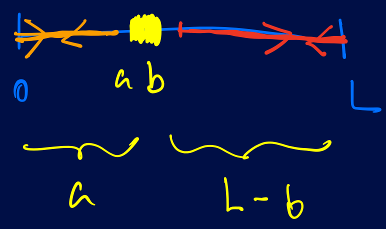
The dynamics of the characteristic lines travelling towards the left and right and bouncing back on the boundary shows that the wave equation generates a periodic dynamics, with time-period $2L$. This decomposition of solutions

$$y(x,t) = \underbrace{f(x+t)}_{\text{transport} \nwarrow} + \underbrace{g(x-t)}_{\text{transport} \nearrow}$$

(which are particular instances of more general transport equations $f_t + v \cdot \nabla_x f = 0$) is a direct consequence of the decomposition of the d'Alembert operator

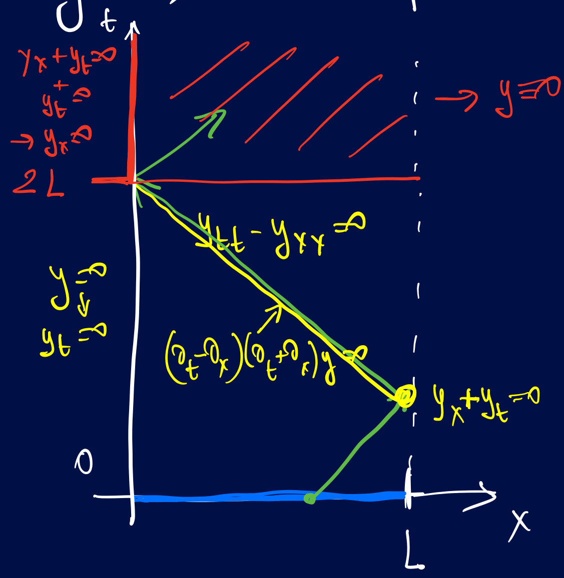
$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x).$$

The dynamics of characteristics shows that, even if the damping region is localized in a sub-interval (a,b) of the space-interval $(0,L)$, all rays will get into the damping region in time $2\max(a, b-L)$.



TRANSPARENT OR PERFECT BOUNDARY CONDITIONS

The construction of effective damping boundary conditions for wave processes (acoustic waves, elasticity, Maxwell, Schrödinger...) is a relevant control theoretical problem. Its applications are numerous and its analysis can get sophisticated since an in-depth understanding requires the use of tools of microlocal analysis. Other applications involve the local numerical approximation for waves in large media. In the one-dimensional case the analysis is much simpler due to the d'Alembert formula.



$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < L, \ t > 0 \\ y = 0, \ x = 0, & t > 0 \\ y_x + y_t = 0, & x = L, \ t > 0 \end{cases}$$

Note that this is a dissipative wave problem. While the Dirichlet boundary condition $y=0$ is conservative at $x=0$, the Neumann like condition

$$y_x = -y_t \quad x = L, \ t > 0$$

is dissipative since

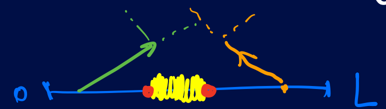
$$E(t) = \frac{1}{2}\int_0^L \left(|y_t|^2 + |y_x|^2\right) dx \rightarrow \frac{dE(t)}{dt} = -|y_t(L,t)|^2.$$

The dissipative boundary condition $y_x + y_t = 0$ at $x = L$ is said to be transparent or perfect since waves when reaching $x=L$ travel toward the outside medium. Due to this the solutions vanish completely after time $t = 2L$. This generates a time-irreversible semigroup. The dissipativity is assured for more general damping terms at $x=L$

$$y_x = -y_t, \ x = L \rightarrow y_x = -\alpha y_t, \ x = L$$

for $\alpha > 0$. But, while the exponential decay of the energy can be guaranteed for any $\alpha > 0$, the fact that solutions reach the equilibrium $y \equiv 0$ in finite-time only holds when $\alpha = 1$.

This very dissipative effect is the one that explains the effectiveness of localized damping too



The two extremes of the boundary of the localized damping patch act as energy absorvers, the same as the dissipative boundary conditions do.

From an analytical perspective, the fact that

$$y_{tt} - y_{xx} + 1_\omega y_t = 0$$

leads to the exponential decay of the energy can be easily proved observing that, since

$$\frac{d}{dt} E(t) = -\int_\omega |y_t|^2 dx$$

it is sufficient to show that, for some $T > 0$, $\exists\, C > 0$ s.t.

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dx dt$$

and this can be proved observing that

$$y_{tt} - y_{xx} = 0 \iff y_{xx} - y_{tt} = 0$$

which allows to use side-wise energy propagation arguments and show the inequality above for $T > 2\max(a, L-b)$.

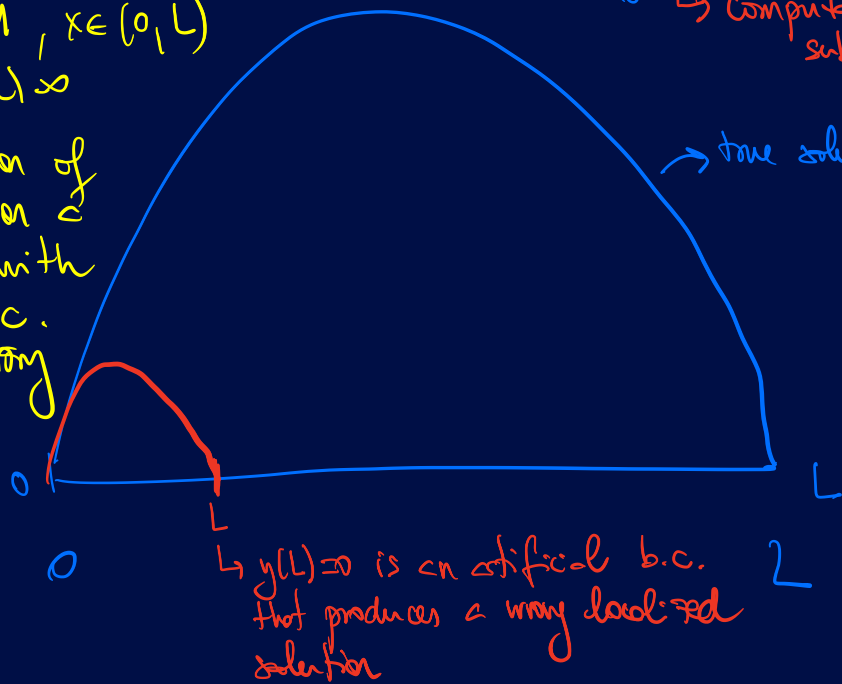
On the use of transparent boundary conditions in numerical approximation of waves.

Achieving a high fidelity numerical approximation of a wave model evolving on a very large domain induces a very high computational cost. It is then natural to analyze whether computations can be implemented locally, concentrating the simulations on the computational subdomain. This is however a complex matter. As we have seen waves propagate in space-time, not remaining confined on a given region. On the other hand, localizing computations on a given subdomain requires to impose boundary conditions on its boundary. But these boundary conditions may have the effect of artificially modifying the behavior of solutions.

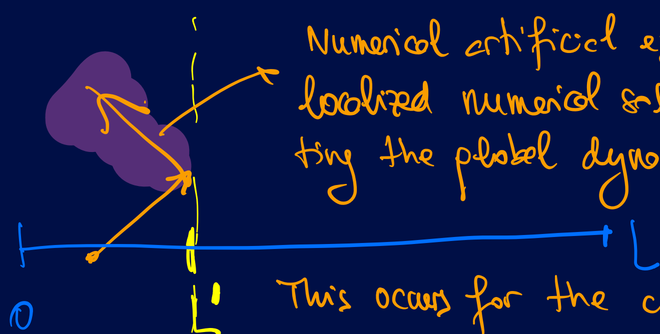
An example:

$$\begin{cases} -y_{xx}=1, & x\in(0,L) \\ y(0)=y(L)=0 \end{cases}$$

The localization of computations on a subdomain with Dirichlet b.c. produces a wrong approximation.



When considering the wave equation $y_{tt}-y_{xx}=0$ an artificial b.c. produces a spurious reflection effect.



Numerical artificial effect → spurious localized numerical solution, non reflecting the global dynamics.

This occurs for the classical b.c. such as Dirichlet and Neumann. We therefore need to employ transparent b.c.

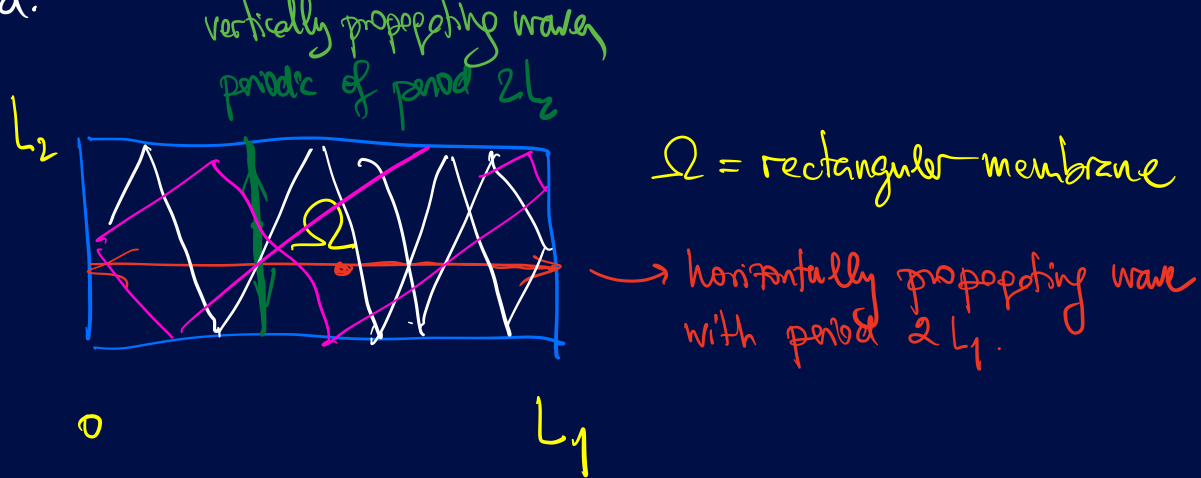
Transparent boundary conditions allow to confine wave computations into subdomains, without introducing spurious numerical components generated by the interaction of waves with the artificial computational boundary.

While transparent boundary conditions are easy to derive for 1-d waves when dealing with the d'Alembert operator $\partial_t^2 - \partial_x^2$, the method is hard to extend to variable coefficients, multi-dimensional models, etc. The method of Perfectly Matching Layers (PML), originally introduced by Bamberger, is much more flexible and provides an adequate solution to the problem in a very broad context.

It is important to observe that even the scalar wave equation generates a very complex dynamics that, at high frequencies, can only be understood by a careful analysis of the propagation properties of the bicharacteristic rays. This leads to dynamical systems behaving as mathematical billiards.

Ω

In particular, we observe that the time-periodic nature of the solutions of the 1-d d'Alembert equation fails in the simplest multi-d domains like the rectangle in 2-d.



[$f_1(t)$ τ_1 - periodic

$f_2(t)$ τ_2 - periodic

Both functions f_1 and f_2 are mutually periodic of period τ if and only if

$\tau_1 / \tau_2 \in \mathbb{Q}$.

As a conclusion, the wave equation in a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ cannot be periodic when $L_1 / L_2 \notin \mathbb{Q}$. Note however that the overall dynamics of the billiard is much more complex than the one shown by these two types of rays.]

PERFECTLY MATCHING LAYERS

Introduced by Bamberger aims to replace the difficult problem of designing transparent boundary conditions for multi-dimensional or complex wave processes, by adding a thin extra artificial layers where waves can penetrate to be damped, so that in this way they do not produce any spurious reflection.

$\tilde{\Omega} = \Omega \cup \omega$

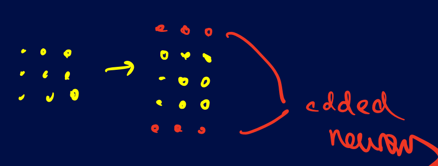
Transparent b.c. risk generating artificial bouncings

Transparent boundary conditions need to handle all possible incoming rays. PML does it much more naturally, letting waves penetrate an added layer of domain all around the boundary where waves penetrate, being damped, without producing any artificial bouncing effects

Roughly the idea is to replace the wave problem in Ω with boundary conditions that will hardly reproduce the behavior on a larger domain, by the damped wave equation in the extended domain $\tilde{\Omega} = \Omega \cup \omega$, ω being the added layer, in such a way that a damping term localized in the added subdomain ω assures that the added layer does not generate any spurious effects

$$y_{tt} - \Delta y + 1_{\omega} y_t = 0 \quad \text{in } (\Omega \cup \omega) \times (0,T).$$

[This idea is still to be explored in the context of Deep NN where the addition of extra neurons with dissipative effects could have a positive influence on the original network.



Composite materials / Homogenization

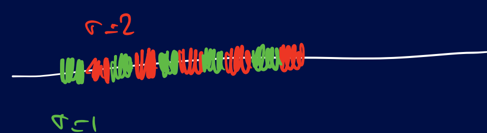
The addition of the extra layer in the PML has to be done carefully since the behavior of PDE solutions in heterogeneous media can become unexpectedly complex.

Homogenization Theory allow to describe the behaviour of composite media achieved out of the mixing of two or more different media. The limiting behaviour is often not the one one could anticipate. Consider for instance

$$-\left(\sigma(x/\varepsilon)\, u_{\varepsilon,x}\right)_x = 1 \quad , x \in \mathbb{R}$$

as $\varepsilon \to 0$, $\sigma(\cdot)$ being a 2-periodic function defined out of the mixing of two materials with material properties $\sigma = 1$ and $\sigma = 2$ respectively.

$$\sigma(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2. \end{cases}$$



One can observe the (unexpected) averaging or homogenization effect emerging when analyzing the limit behavior of solutions $u_\varepsilon(x)$ as $\varepsilon \to 0$. We perform the computation without paying attention to the boundary conditions since these are mainly local phenomena.

Integrating in x the equation

$$-\left(\sigma\left(\tfrac{x}{\varepsilon}\right) u_x\right)_x = 1$$

we get

$$-\sigma\left(\tfrac{x}{\varepsilon}\right) u_{\varepsilon,x} = x$$

and therefore

$$u_{\varepsilon,x} = \frac{-x}{\sigma(x/\varepsilon)}$$

In view of the periodicity of $\sigma(x/\varepsilon)$, $1/\sigma(x/\varepsilon)$ is periodic as well and its weak (L^∞-weak * limit) is given by the average of $1/\sigma$:

$$\frac{1}{\sigma(x/\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^2 \frac{1}{\sigma}\,ds = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}.$$

The weak limit $v = v(x)$ in H^1_{loc} of $u_\varepsilon(x)$ then is characterized by the equation

$$v_x(x) = -\frac{3}{4}x$$

which can be rewritten as

$$-\left(\frac{4}{3}v_x\right)_x = 1.$$

We therefore observe a nonlinear averaging effect! The effective limit conductivity is $4/3$ and not the average of σ that yields $\bar{\sigma} = \frac{1}{2}(1+2) = \frac{3}{2}$.

We could also consider the example of the 3-periodic function

$$\sigma(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 3. \end{cases}$$

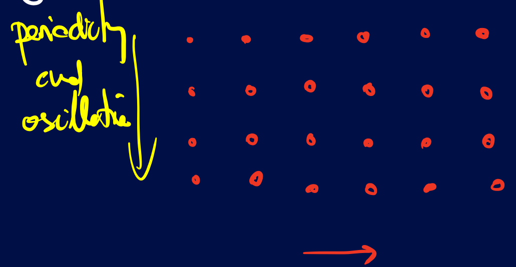
Then, the weak limit of $1/\sigma(x/\varepsilon)$ would be $\int_0^3 \frac{1}{\sigma}$, i.e.

$$\int_0^3 \frac{1}{\sigma}\,ds = \frac{1}{3}\left(1+\frac{2}{2}\right) = \frac{2}{3}.$$

This would lead to a composite material with effective coef. $3/2$

Rapid oscillations on coefficients can lead to different surprising phenomena. Floquet theory permits to analyze the dynamics of ODEs with periodic coefficients. Homogenization problems make sense in that context too.

The behavior of Deep Neural Networks in the homogenization regime is a interesting topic too. Rapid oscillations of the NN can be present in two different ways: In a time-like manner along the consecutive layers by also on each layer along the accumulated neurons.



→ periodicity and oscillations matter

Similar issues arise also for dissipative wave equations with time-dependent coefficients. For instance the dissipative wave equation with time-dependent density

$$\rho(t)\, y_{tt} - \Delta y + y_t = 0$$

(which, in some sense, could be compared to a Neural ODE

$$x'(t) = W(t)\,\sigma(A(t)\,x(t) + b(t))$$
)
can be unexpectedly difficult to analyze.

In the particular case where the density is constant ($\rho(t) \equiv 1$) we have shown the exponential decay of the energy

(damped wave eqn) $y_{tt} - \Delta y + y_t = 0$

(damped energy) $E(t) = \frac{1}{2}\int_\Omega \left(|\nabla y|^2 + |y_t|^2\right) dx$

(exponential decay) $E(t) \leq C e^{-\alpha t} E_0 \rightarrow [0, T_\varepsilon]; \; E(T) \leq C e^{-\alpha T} E_0$

$E \mapsto E_\varepsilon = E + \varepsilon \int_\Omega y y_t \, dx$ (perturbed energy)

$$\frac{dE}{dt} = -\int_\Omega |y_t|^2 dx$$ (energy dissipation law

variable density damped wave eqn $\rho(t) y_{tt} - \Delta y \boxed{+ y_t} = 0$

Corresponding energy $E(t) = \frac{\rho(t)}{2}\int_\Omega |y_t|^2 dx + \frac{1}{2}\int_\Omega |\nabla y|^2 dx$

$$\frac{dE}{dt}(t) = \frac{\rho'(t)}{2}\int_\Omega |y_t|^2 dx + \underbrace{\rho(t)\int_\Omega y_t y_{tt} + \int_\Omega \nabla y \nabla y_t}_{-\int_\Omega y_t^2 dx}$$

$$= -\left(1 - \frac{\rho'(t)}{2}\right)\int_\Omega |y_t|^2 dx$$

The dissipative nature of the system is only guaranteed when $\rho'(t) \leq 2$.

The added velocity feedback term does not have a clear dissipative effect unless $\rho'(t) \leq 2$.

The same happens for the harmonic oscillator

- $x'' + x + x' = 0$ (damped harmonic oscillator)
- $x'' + Ax + x' = 0$ (damped system of harmonic oscillators)
- $y_{tt} - \Delta y + y_t = 0$ (damped wave equation)

When the mass m is independent of time the dissipate effect is rather explicit

$$m x'' + x + x' = 0 \quad \sim \quad m x'' + x = -x'$$

$$e_m(t) = \frac{m}{2}|x'(t)|^2 + \frac{1}{2}|x(t)|^2 \quad \text{(energy)}$$

$$\frac{de}{dt} = -|x'|^2 \leq 0 \quad \text{(energy dissipation)}$$

But the dissipative effects are much less obvious for variable masses:

$$m(t)\, x'' + x + x' = 0$$

$$e(t) = \frac{m(t)}{2}|x'(t)|^2 + \frac{1}{2}|x(t)|^2$$

$$\frac{d}{dt}e(t) = \frac{m'(t)}{2}|x'(t)|^2 + m(t)\, x'(t)\, x''(t) + x(t)\, x'(t)$$

$$= -\left(1 - \frac{m'(t)}{2}\right)|x'(t)|^2$$

Similarly as for the wave equation the dissipative effect of this energy is only explicitly observed when $m'(t) \leq 2$.

Note however that, even for the wave equation, modifying the natural energy to build the augmented energy $E_\varepsilon(t)$ allowed to better observe the dissipative nature of the system.

Therefore the question arises of how to build optimal Lyapunov functionals to better identify the dissipative nature of finite-dimensional systems. This is now being investigated using Neural Network techniques, by Lars Grüne (Bayreuth) and coworkers.

There are other interesting connections between the the theory of damped harmonic oscillators and Deep Neural Networks (DNN)

We have seen that the ODE interpretation of ResNets leads to the so-called Neural ODEs:

$$x'(t) = W(t)\,\sigma\big(A(t)\,x(t) + b(t)\big)$$

σ being, for instance, the ReLU function ReLU

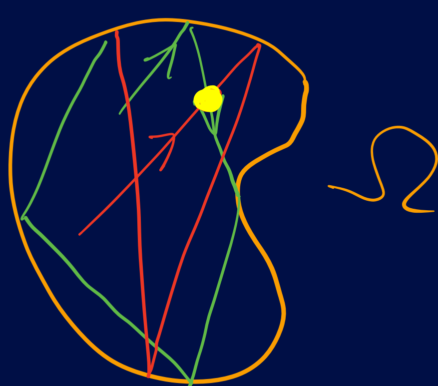
The so-called Momentum ResNets or their ODE counterpart are of the form are obtained adding a second order inertia term to the Neural ODE:

$$\varepsilon x''(t) + x'(t) = W(t)\,\sigma(A(t)x(t) + b(t)).$$

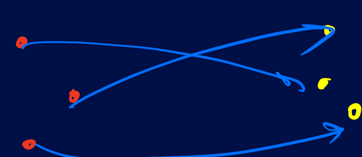
The state of this new augmented Neural ODE becomes

$$(x(t), x'(t)).$$

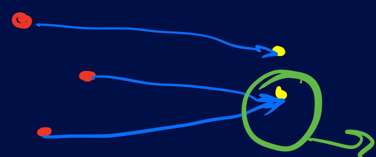
This, second order, continuous in time version of Deep Neural Networks allows that different data or trajectories get to the same final destination, at the same time $t=T$, but with different velocities, not to contradict the uniqueness property of the solutions of the corresponding Cauchy problem. This is the same reason that allows for the bicharacteristic rays of the mathematical billiard describing the propagation of waves.



First order Neural ODEs



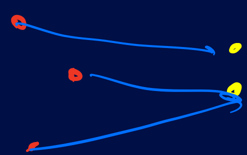
Data can be classified to reach different destinations



Impossible!

Data cannot be classified so that two of them get to the same destination. This violates the uniqueness property of the backward ODE.

Second order momentum ODEs



Two different data can be classified to get to the same exact destination, but with different velocities.

The limit process

Using the classical asymptotic and singular limit techniques for ODEs one can pass to the limit as $\varepsilon \to 0$ from

$$\begin{cases} \varepsilon x'' + x' = W(t)\,\sigma(A(t)\,x(t) + b(t)) \\ x(0) = x_0, \quad x'(0) = x_1. \end{cases}$$

Of course, as $\varepsilon \to 0$, the initial velocity is lost:

$$\begin{cases} x' = W(t)\,\sigma(A(t)\,x(t) + b(t)) \\ x(0) = x_0. \end{cases}$$

Of course, when $\varepsilon \sim 1$ or $\varepsilon \gg 1$, the behaviour of Momentum Neural ODEs can be quite different from the first order Neural ODE. This can be clearly seen in the context of the damped harmonic oscillator when comparing

$$\varepsilon x'' + x + x' = 0$$

and

$$x' + x = 0.$$

$$\left[\varepsilon x'' + x + x' = 0 \quad \to \quad \varepsilon\lambda^2 + 1 + \lambda = 0\right.$$

$$\lambda = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$$

It is immediately seen that the dynamics changes depending on whether $4\varepsilon > 1$ or $4\varepsilon < 1$.$\Big]$

Momentum ResNets and Memory effect.

The effect of the second order inertia term in Momentum ResNets can also be interpreted as a memory term. To illustrate this fact we may consider the damped harmonic oscillator

$$\varepsilon x'' + x + x' = 0$$

that can be rewritten as

$$\left(x'' + \frac{1}{\varepsilon}x'\right) + \frac{1}{\varepsilon}x = 0$$

so that we get

$$\left(e^{t/\varepsilon}x'\right)' + \frac{e^{t/\varepsilon}x}{\varepsilon} = 0$$

After integration we get

$$e^{t/\varepsilon}x'(t) + \frac{1}{\varepsilon}\int_0^t e^{s/\varepsilon}x(s)\,ds = x^1$$

or, equivalently,

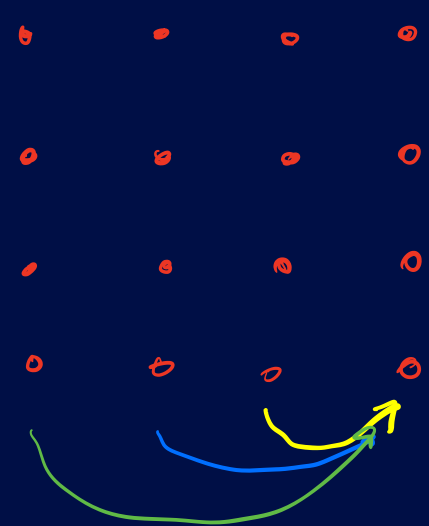
$$x'(t) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}x(s)\,ds = e^{-t/\varepsilon}x_1.$$

In the particular case where $x_1 = 0$ we get

$$x'(t) + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)/\varepsilon}x(s)\,ds = 0.$$

This motivates the possible use of Neural ODEs with memory terms (ongoing work in collaboration with Elise Affili and Domènec Ruiz-Balet). They allow for instance, not only to drive trajectories to a given target but also assure that they track approximately a given trajectory:

$$x'(t) + W(t)\,\sigma\left(A(t)\,x(t)+b(t),\ \underbrace{\int_0^t \left[\tilde{A}(t,s)\,x(s) + c(t,s)\right] ds}_{\text{memory term.}}\right) = 0$$



Of course the memory term can also be implemented in time-discrete Deep Neural Networks. The principle is the same as in the gradient descent algorithms where one can use past information to accelerate the convergence.

[From the steepest descent algorithm $x^{k+1} = x^k - \rho \nabla J(x^k)$ to the variant involving delay $x^{k+1} = x^k - \rho \nabla J(x^{k-1})$.]

Viscoelasticity and memory effects.

Memory effects are also typical in continuum Mechanics.

Consider, for instance, a model in elasticity with viscous damping

$$y_{tt} - \sigma \Delta y - \nu \Delta y_t = 0.$$

It can also be interpreted as a model with memory. Indeed, integrating in time leads to

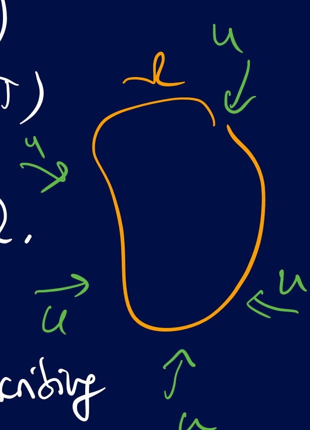
$$y_t - \nu \Delta y - \sigma \Delta \int_0^t y(s)\,ds = 0,$$

which can be viewed as a heat equation with memory effects on the viscosity.

BOUNDARY CONTROL OF THE HEAT EQUATION.

Consider the following multi-d wave equation with boundary control

$$\begin{cases} y_{tt} - \Delta_x y = 0 & \Omega \times (0,T) \\ y = u & \partial\Omega \times (0,T) \\ y(0) = y_0, \; y_t(0) = y_1 & \Omega. \end{cases}$$



$y = y(x,t)$ is the solution or state describing elastic vibrations or acoustic waves

$u = u(x,t)$ is the control

What are the new dynamics we can generate by a suitable choice of the boundary controls $u(x,t)$?

Can we for instance find $u = u(x,t)$, localized on the boundary $\partial\Omega \times (0,T)$, so that

$$y(x,T) \equiv y_t(x,T) \equiv 0.$$

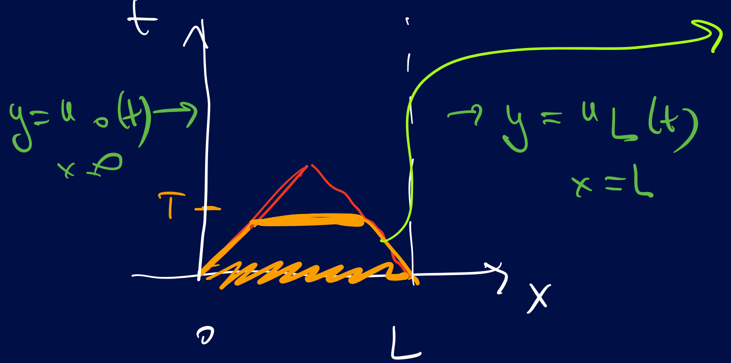
When doing so we are trying to drive the system to equilibrium in time $t = T$ by means of an open loop control $u(x,t)$.

A new phenomenon: Control in large time

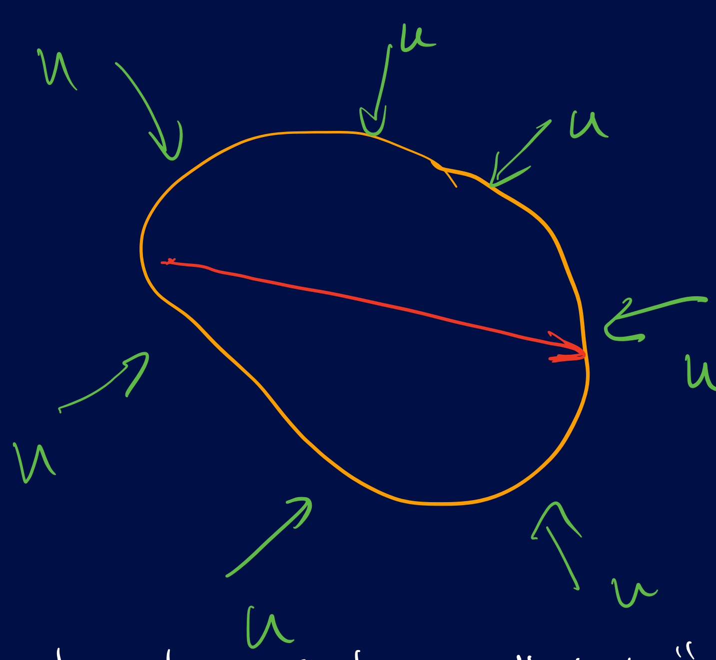
For wave equations the velocity of propagation is finite and, therefore, such a control property requires the time-horizon for control, T, to be large enough.

This is new with respect to the (A,B) finite-dimensional systems, because in that context when the system is controllable the control target can be achieved in any time $T > 0$.

The fact that $T > 0$ needs to be large enough to successfully control the wave equation from the boundary can be easily seen in 1-d by means of the d'Alembert formula

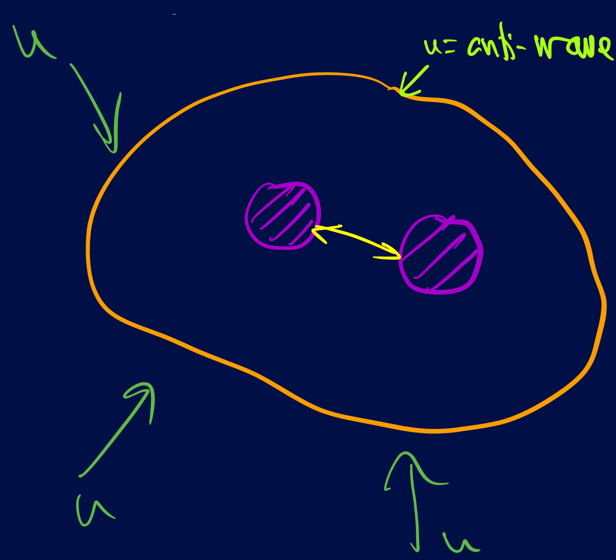


The solution $y(x,t)$ is fully determined by the initial data inside the characteristic triangle and it is completely independent of the boundary controls $u_0(t)$ and $u_L(t)$.



In the multi-dimensional setting the time of control needs to be at least as large as the time the bicharacteristic rays of the mathematical billiard need to get to the boundary.

Controls act as "anti-waves" that "kill" the waves when reaching the boundary. Their dynamics cannot be affected before they reach the boundary.



Perforated membrane.

A membrane, perforated by two or more holes can generate trapped rays within the obstacles, making it impossible their control from the exterior boundary.

Dual observability property.

The adjoint wave equation reads

$$
\begin{cases}
\varphi_{tt} - \Delta_x \varphi = 0 & \Omega \times (0,T) \\
\varphi|_{\partial\Omega} = 0 & \\
\varphi(x,T) = \varphi_{0,T},\ \varphi_t(x,T) = \varphi_{1,T} & \text{in } \Omega
\end{cases}
$$

Reminder: State and adjoint state system.

$$x' + Ax = 0 \to -\varphi' + A^*\varphi = 0.$$

The energy of solutions of the adjoint system is conserved:

$$E(t) = \frac{1}{2}\int_\Omega |\nabla \varphi(x,t)|^2\,dx + \frac{1}{2}\int_\Omega |\varphi_t(x,t)|^2\,dx$$

$$E(t) = E_T = \frac{1}{2}\int_\Omega |\nabla \varphi_{0,T}|^2\,dx + \frac{1}{2}\int_\Omega |\varphi_{1,T}|^2\,dx$$

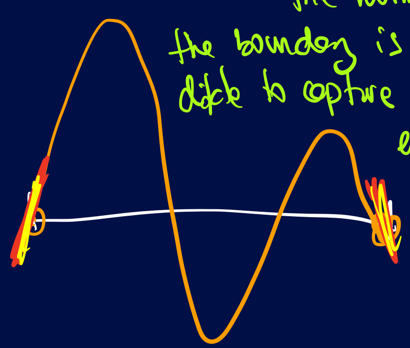
The boundary controllability of the wave equation turns out to be equivalent to the following observability inequality for the adjoint system

$$E(T) \le C\int_0^T\int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt, \quad \forall \varphi \text{ solution of the adjoint system}$$

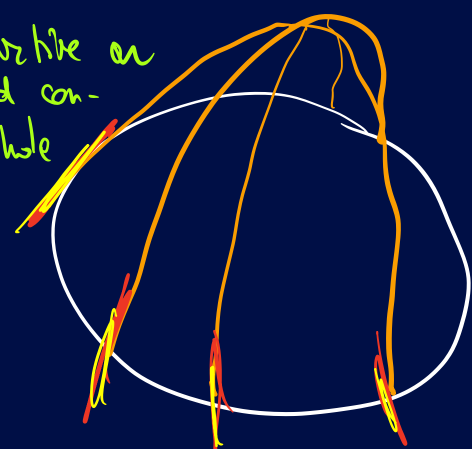
This inequality ensures that in time $T>0$ all the energy of the vibrations can be estimated in terms of the energy concentrated on the boundary.

$\vec{\nu}(x)$ = outer unit normal vector

$$\varphi\big|_{\partial\Omega\times(0,T)} = 0 \Rightarrow \partial_t\varphi = \nabla_\tau\varphi = 0 \text{ on } \partial\Omega\times(0,T) \ \Big/\ \nabla\varphi = \frac{\partial\varphi}{\partial\nu}\vec{\nu}\,\Big)$$



The normal derivative on the boundary is a good candidate to capture the whole energy



OBSERVABILITY PROBLEM Given a domain $\Omega \subset \mathbb{R}^d$, bounded, find a time $T>0$ s.t. $\exists\, C_T > 0$

observability inequality

$$E(T) \le C_T\int_0^T\int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt$$

$\forall \varphi$ soln of the adjoint pbm.

$\varphi = \varphi(x,t)$ being a solution of the adjoint, it represent a free vibration without control.

Why the observability inequality suffices?

$$\left.\begin{aligned} &y_{tt} - \Delta y = 0 \\ &y = u \\ &y(0) = y_0,\ y_t(0) = y_1 \end{aligned}\right\} \longrightarrow \begin{array}{l}\text{Target} \\ y(T) \equiv y_t(T) \equiv 0.\end{array}$$

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 \\ \varphi|_{\partial\Omega} = 0 \\ \varphi(T) = \varphi_{0,T},\ \varphi_t(T) = \varphi_{1,T} \end{cases}$$

The contribution of time $t=T$ in this integration by parts vanishes when the state $y = y(x,t)$ reaches equilibrium: $y(T) \equiv y_t(T) \equiv 0$.

$$0 = \int_\Omega (y_t\varphi - y\varphi_t)\,dx \Big|_0^{\cancel{T}} + \int_0^T\int_{\partial\Omega} u\,\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt = 0$$

$\Rightarrow$ The control $u = u(x,t)$ successfully controls the system to $y(T) \equiv y_t(T) \equiv 0$ iff $u(x,t)$ is such that the following identity holds for all solutions of the adjoint system $\varphi = \varphi(x,t)$:

$$0=\int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial\nu}\,d\sigma dt-\int_\Omega y_1\varphi(0)\,dx+\int_\Omega y_0\varphi_t(0)\,dx$$

Accordingly, to find the control, it is sufficient to minimize the functional

$$J(\varphi_{0,T},\varphi_{1,T})=\frac{1}{2}\int_0^T\int_{\partial\Omega}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt-\int_\Omega\left(y_1\varphi(0)-y_0\varphi_t(0)\right)dx$$

over the energy space $H=H_0^1(\Omega)\times L^2(\Omega)$ (which is a Hilbert space).

Indeed, if $(\hat\varphi_{0,T},\hat\varphi_{1,T})$ is a minimizer

$$J(\hat\varphi_{0,T},\hat\varphi_{1,T})=\min_{(\varphi_{0,T},\varphi_{1,T})\in H} J(\varphi_{0,T},\varphi_{1,T})$$

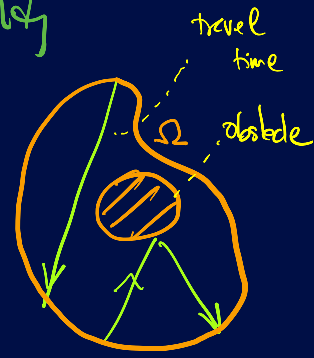
then

$$\hat u=\partial\hat\varphi/\partial\nu$$

is a successfull control.

To assure that the functional J has a minimizer we need to assure its coercivity and this is precisely the role of the observability inequality.

Proof of the observability inequality



$$\begin{cases} \varphi_{tt} - \Delta_x \varphi = 0 & \Omega \times (0,T) \\ \varphi|_{\partial\Omega} = 0 & \\ \varphi(T) = \varphi_{0,T}, \ \varphi_t(0) = \varphi_{1,T} & \Omega \end{cases}$$

$$\boxed{E \leq C \int_0^T \int_{\partial\Omega} \left|\frac{\partial \varphi}{\partial \nu}\right|^2 d\sigma\, dt.}$$

This inequality plays an important role in the theory of Inverse Problems too, when trying to detect interior properties of the wave-medium out of boundary measurement.

Proof of the observability inequality: Multiplier technique.

We first use the multiplier:

$$0 = \int_0^T \int_\Omega (\varphi_{tt} - \Delta \varphi)\, x \cdot \nabla \varphi \, dx\, dt.$$

$$\left.\begin{aligned} x &= (x_1 - x_d) \in \Omega \\ \nabla \varphi &= \left(\frac{\partial \varphi}{\partial x_1} \ldots \frac{\partial \varphi}{\partial x_d}\right) \end{aligned}\right\} \quad x \cdot \nabla \varphi = \sum_{j=1}^{d} x_j \frac{\partial \varphi}{\partial x_j}.$$

$$\int_0^T \int_\Omega \varphi_{tt}\, x \cdot \nabla \varphi \, dx\, dt - \int_0^T \int_\Omega \Delta_x \varphi \; x \cdot \nabla \varphi \, dx\, dt = 0.$$

- $\int_0^T \int_\Omega \varphi_{tt}\, x\cdot\nabla_x \varphi \, dx\, dt$ $\qquad x\cdot\nabla_x\varphi = |x| \frac{\partial \varphi}{\partial r}$

$$= -\int_0^T\int_\Omega \varphi_t \, x\cdot\nabla\varphi_t \, dx\, dt + \int_\Omega \varphi_t \, x\cdot\nabla\varphi \, dx \Big|_0^T$$

$$= -\int_0^T\int_\Omega \frac{x}{2}\cdot\nabla\left[(\varphi_t)^2\right] dx\, dt + \quad ''$$

$$= \underbrace{\frac{d}{2}\int_0^T\int_\Omega \varphi_t^2 \, dx\, dt}_{\text{kinetic energy}} + \quad ''$$

- $-\int_0^T\int_\Omega \Delta\varphi \, x\cdot\nabla\varphi \, dx\, dt$

$$= \int_0^T\int_\Omega \nabla\varphi\cdot\nabla(x\cdot\nabla\varphi)\, dx\, dt - \int_0^T\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \, x\cdot\nabla\varphi \, d\sigma\, dt$$

$x\cdot\nabla\varphi \big|_{\partial\Omega} = ?$

$\varphi\big|_{\partial\Omega} = 0 \rightarrow \nabla\varphi\big|_{\partial\Omega} = \cancel{\nabla_\tau \varphi \, \tau} + \frac{\partial\varphi}{\partial\nu}\nu$

$\rightarrow x\cdot\nabla\varphi\big|_{\partial\Omega} = (x\cdot\nu)\frac{\partial\varphi}{\partial\nu}.$

$$= \int_0^T \int_\Omega \nabla\varphi \cdot \nabla (x \cdot \nabla\varphi) \, dx\,dt - \int_0^T \int_{\partial\Omega} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt$$

$$= \int_0^T \int_\Omega \partial_j\varphi \, \partial_j (x_k \partial_k \varphi) \, dx\,dt \quad - \quad ''$$

$$= \int_0^T \int_\Omega \partial_j\varphi \left(\delta_{jk} \partial_k \varphi + x_k \partial^2_{jk}\varphi \right) dx\,dt \quad - \quad ''$$

$$= \int_0^T \int_\Omega |\nabla\varphi|^2 dx\,dt + \int_0^T \int_\Omega \partial_j \varphi \, x_k \, \partial^2_{jk}\varphi \, dx\,dt - \int_0^T \int_{\partial\Omega} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt$$

$$= \int_0^T \int_\Omega |\nabla\varphi|^2 dx\,dt + \int_0^T \int_\Omega x \cdot \nabla \left| \frac{|\nabla\varphi|^2}{2} \right| dx\,dt +$$

$$= \left(1 - \frac{d}{2}\right) \underbrace{\int_0^T \int_\Omega |\nabla\varphi|^2 dx\,dt}_{\text{potential energy}} + \underbrace{\frac{1}{2} \int_0^T \int_{\partial\Omega} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt}_{\text{observed boundary energy}} - ''$$

$$0 = \frac{d}{2} \int_0^T \int_\Omega |\varphi_t|^2 dx\,dt + \left(1 - \frac{d}{2}\right) \int_0^T \int_\Omega |\nabla\varphi|^2 dx\,dt$$

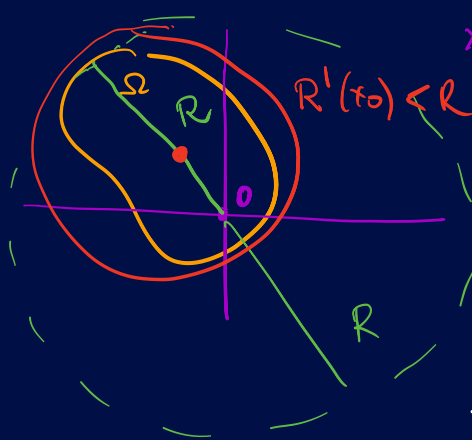
$$- \frac{1}{2} \int_0^T \int_{\partial\Omega} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt + \int_\Omega \varphi_t \, x \cdot \nabla\varphi \, dx \Big|_0^T$$

$$\frac{d}{2}\int_0^T\int_\Omega |\varphi_t|^2\,dx\,dt + \left(1-\frac{d}{2}\right)\int_0^T\int_\Omega |\nabla\varphi|^2\,dx\,dt + \int_\Omega \varphi_t\, x\cdot\nabla\varphi\,dx\Big|_0^T$$

$$= \frac{1}{2}\int_0^T\int_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\,dt$$

$$\le \frac{R}{2}\int_0^T\int_{\partial\Omega}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\,dt.$$

$\left[\ |x\cdot\nu| \le \|x\|\,\|\nu\| = \|x\| \le R = \max_{x\in\bar\Omega}\|x\|\right.$



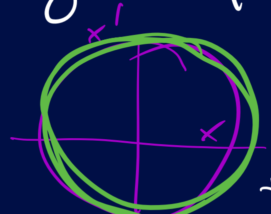
$x\cdot\nabla\varphi \to (x-x_0)\cdot\nabla\varphi$

The center $x=0$ of this inequality can be translated to any point x_0. A suitable choice of x_0 leads to different subdomains of the boundary for observation and also to different observation times.

Equipartition of energy.

The fact that the energy of solutions of the wave equation is essentially equidistributed and balanced between the kinetic and potential can be observed for the harmonic oscillator: $x'' + x = 0$. Multiplying the equation by x' we get that the energy

$$e(t) = \frac{1}{2}|x'|^2 + \frac{1}{2}|x|^2$$



is conserved. Of course, at any fixed t, there is no relation between the kinetic component $|x'|^2$ and the potential one $|x|^2$, as can be seen in the phase portrait of trajectories. But both components get better balanced when integrating in time. Actually, multiplying the equation by $x(t)$ and integrating in $\tau_1 < t < \tau_2$ we get

$$-\int_{\tau_1}^{\tau_2} |x'|^2\,dt + \int_{\tau_1}^{\tau_2} |x|^2 = x(\tau_2)x'(\tau_2) - x(\tau_1)x'(\tau_1).$$

$$\underbrace{\int_{\tau_1}^{\tau_2} x^2\,dt}_{\text{potential energy}} \; \underbrace{+\, x x'\Big|_{\tau_1}^{\tau_2}}_{\text{Reminder}} \; = \underbrace{\int_{\tau_1}^{\tau_2} |x'|^2\,dt}_{\text{kinetic energy}}.$$

The same computation can be performed for the wave equation $\varphi_{tt} - \Delta\varphi = 0$.

Multiplying by φ we have:

$$\int_0^T\int_\Omega \varphi_{tt}\varphi\,dx\,dt - \int_0^T\int_\Omega \Delta\varphi\,\varphi\,dx\,dt = 0$$

$$-\int_0^T\int_\Omega |\varphi_t|^2\,dx\,dt + \int_\Omega \varphi_t\varphi\,dx\Big|_0^T + \int_0^T\int_\Omega |\nabla\varphi|^2\,dx\,dt = 0$$

$$\int_0^T\int_\Omega \left(|\varphi_t|^2 - |\nabla\varphi|^2\right)dx\,dt = \underbrace{\int_\Omega \varphi_t\varphi\,dx\Big|_0^T}_{\text{Reminder term, which is compact in the energy space.}}$$

Back to the boundary observability inequality.

$$\underbrace{\frac{1}{2}\int_0^T\int_\Omega \left(|\varphi_t|^2 + |\nabla\varphi|^2\right)dx\,dt}_{TE_0} + \cancel{\frac{d-1}{2}\int_0^T\int_\Omega \left(|\varphi_t|^2 - |\nabla\varphi|^2\right)dx\,dt}$$

$$+ \int_\Omega \varphi_t\left(x\cdot\nabla\varphi + \frac{d-1}{2}\varphi\right)dx\Big|_0^T$$

$$\le \frac{R}{2}\int_0^T\int_{\partial\Omega}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt$$

$$T E_0 + \left|\boxed{\int_\Omega \varphi_t\left(x\cdot\nabla\varphi + \frac{d-1}{2}\varphi\right)dx\,dt\Big|_0^T}\right| \le \frac{R}{2}\int_0^T\int_{\partial\Omega}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt$$

$$\left|\int_\Omega \varphi_t\left(x\cdot\nabla\varphi + \frac{d-1}{2}\varphi\right)dx\Big|_0^T\right| \le M\left(E_0 + E(T)\right) = 2ME_0$$

Cauchy-Schwartz

$$\left|\int_\Omega \varphi_t\left(x\cdot\nabla\varphi + \frac{d-1}{2}\varphi\right)dx\right| \le \|\varphi_t\|_{L^2(\Omega)}\left\|x\cdot\nabla\varphi + \frac{d-1}{2}\varphi\right\|_{L^2(\Omega)}$$

$$\le M\|\varphi_t\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}$$

$$\le \frac{M}{2}\left(\|\varphi_t\|_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)}\right)$$

$$= M\,E(t).$$

We are done!

$$T E_0 - 2ME_0 \le \frac{R}{2}\int_0^T\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt.$$

$T>0$ arbitrary $\qquad M = M(\Omega, d)$

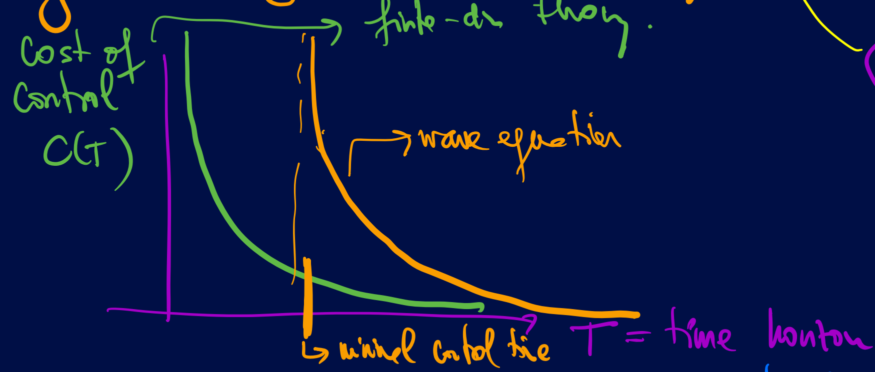
Consequently, if $T > 2M(\Omega, d)$ we have

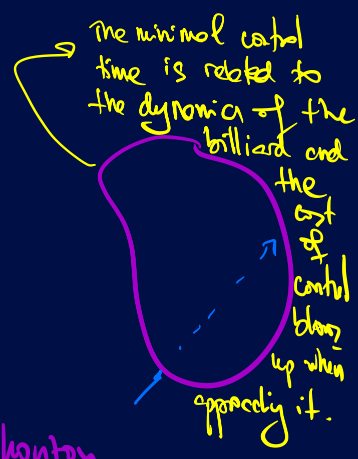
$$\boxed{E_0 \le \frac{R}{4M}\int_0^T\int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma \, dt}$$

Corollary: If $T > 0$ is large enough, with $T = T(\Omega)$ depending on the geometry of Ω, the wave equation is boundary observable and, by duality, boundary controllable as well.

Remark about the time of control.

- The fact that $T > 0$ needs to be large is new with respect to the finite-dimensional dynamical system control theory.





The "waiting time" phenomenon for the control of the wave equation is a new phenomenon that does not arise for finite-dimensional systems.

in The "cost of control", $C(T) = C_T$, is defined as the best ct.

$$\|u\|_{L^2(\partial\Omega\times(0,T))} \leq C_T \|(y_0, y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}$$

C_T = norm of the linear map "i.d. → control"

The fact that the boundary control of the wave equation needs of a minimal waiting-time is intrinsic to the finite-velocity of propagation of the model.

Control on part of the boundary.

The same multiplier techniques allows to show that the controllability can be achieved acting on a suitable subset of the boundary. It suffices to observe that

$$\int_0^T \int_{\partial\Omega} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt$$

$$= \int_0^T \int_{\partial\Omega^+} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt + \int_0^T \int_{\partial\Omega^-} (x\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt$$

$$\partial\Omega = \partial\Omega^+ \cup \partial\Omega^-$$

$$\partial\Omega^+ = \{x\in\partial\Omega : x\cdot\nu > 0\}$$

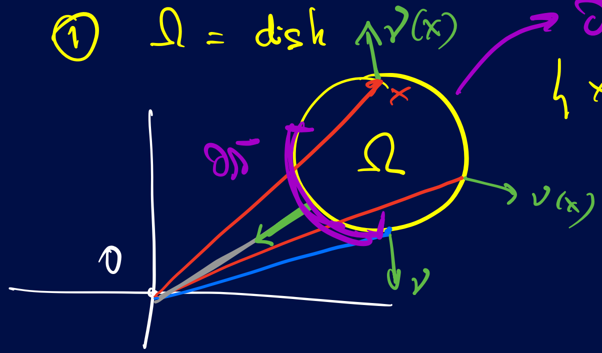
$$\partial\Omega^- = \{x\in\partial\Omega : x\cdot\nu < 0\}.$$

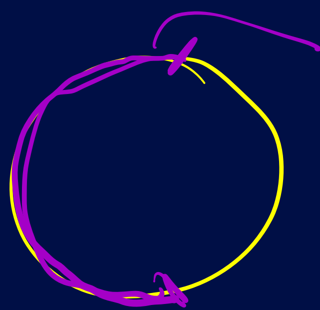
$$\leq R \int_0^T \int_{\partial\Omega^+} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt.$$

Ex. (1) Ω = disk

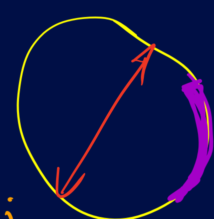
$\partial\Omega^+$

$\{x \in \partial\Omega : x \cdot \nu > 0\}$





$\partial\Omega^+$ is always an arc that contains more than half of the perimeter. The controllability cannot be achieved acting on less than half of the circle since there are diameters that escape of the observation and control



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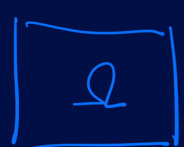
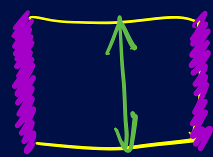


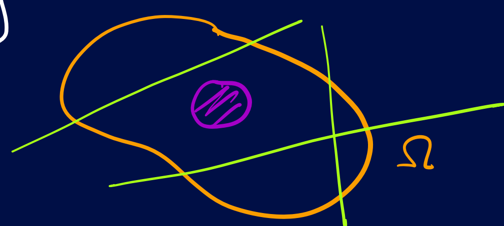
When Ω is a rectangle, depending on where the center x_0 of the multiplier is located, the control can occupy either three sides or two sides of the boundary.



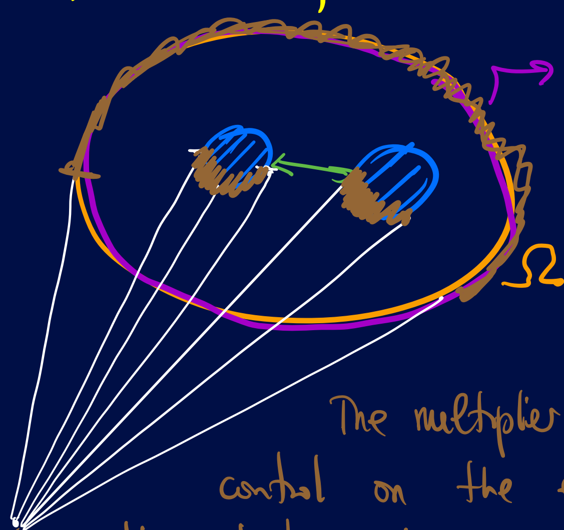
→ Note however that the control on two parallel sides does not suffice since there can be parallel rays escaping the observation.

→ The same occurs if we miss two parallel segments of the three-controlling sides.

Of course, one can observe similar pathologies in the context of Inverse Problems. For instance, when trying to detect a deffect or obstacle inside a domain through boundary measurements along the boundary the obstacle can be missed if the boundary measurements are not exhaustive enough



$\boxed{3}$ Domains with holes



→ The control on the exterior boundary does not suffice because of the ray trapped within the two obstacles.

The multiplier technique leads to areas of control on the exterior boundary but also on the obstacles inside, guaranteeing the control of all waves.

Control interpretation of the observability inequation.

Given a domain Ω, bounded, we have shown the existence of a time $T(\Omega) > 0$ such that the following inequality holds for all solutions of the adjoint equation

$$E_0 \leq C \int_0^T \int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt, \quad \forall \varphi \text{ sol'n of the adjoint system}$$

The same occurs when controlling on subsets of the form $\partial\Omega_+$:

$$E_0 \leq C \int_0^T \int_{\partial\Omega_+} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt, \quad \forall \varphi \text{ solution of the adjoint problem.}$$

Let us now consider the control problem:

$$\begin{cases} y_{tt} - \Delta y = 0 & \Omega \times (0,T) \\ y = \begin{cases} u & \text{on } \partial\Omega^+ \times (0,T) \\ 0 & \text{on } \partial\Omega^- \times (0,T) \end{cases} \\ y(0) = y_0, \ y_t(0) = y_1 & \text{in } \Omega. \end{cases}$$

The goal is to find a control $u = u(x,t)$ s.t.

$$y(x,T) \equiv y_t(x,T) \equiv 0.$$

We know that controllability is guaranteed with controls $u = u(x,t)$ with support on $\partial\Omega^+ \times (0,T)$.

To build the control it suffices to minimize the functional

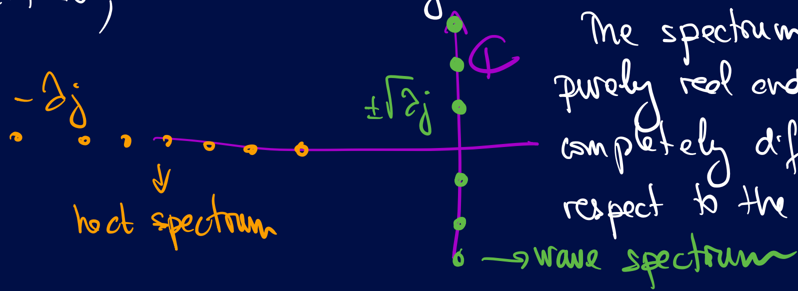
$$J_{\partial\Omega^+}(\varphi_{0,T}, \varphi_{1,T}) = \frac{1}{2}\int_0^T \int_{\partial\Omega^+} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt - \int_\Omega \left(y_1 \varphi(0) - y_0 \varphi_t(0)\right) dx$$

$\rightarrow \left(\hat{\varphi}_{0,T}, \hat{\varphi}_{1,T}\right) \rightarrow$ minimizer $\rightarrow$ control $u = \frac{\partial \hat{\varphi}}{\partial \nu} 1_{\partial\Omega \times (0,T)}$.

THE HEAT EQUATION

$$\begin{cases} y_t - \Delta y = 0 & \Omega \times (0,T) \\ y = u & \partial\Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

Contrarily to the wave equation, it can be controlled to zero, i.e. $y(T) = 0$, in an arbitrarily short time $T > 0$.



The spectrum of the heat equation, purely real and negative shows the completely different dynamics with respect to the wave equation.

The fact that the behaviour of wave and heat equations with respect to boundary control differs so much, suggests that the control problem can be intricate for hybrid systems.

Kannai or transmutation transformation

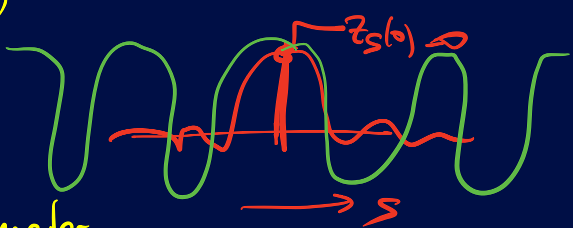
The heat equation is subordinated to the wave equation

heat-like abstract model
$$\begin{cases} Y_t + AY = 0 & , t > 0 \\ Y(0) = Y_0 \end{cases}, \quad Y \in C([0,T]; X), \quad Y_0 \in X$$

- heat equation $A = -\Delta$
- ODE system $X = \mathbb{R}^n$, $A \in M_{n \times n}$

The key assumptions for transmutation are that A = linear and time-independent

wave-like model $\begin{cases} Z_{ss} + AZ = 0, & -\infty < s < \infty \\ Z(0) = Y_0, \quad Z_s(0) = 0 \end{cases}$



s = pseudo-time, parameter.

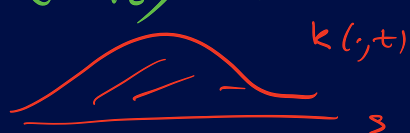
Both solutions $Z(s)$ and $Y(t)$ are related by the transmutation identity

$$Y(t) = \int_{-\infty}^{\infty} W(s)\, k(s,t)\, ds$$

where $k(s,t)$ is the 1-d heat kernel

$$k(s,t) = (4\pi t)^{-1/2} \exp\left(-\frac{s^2}{4t}\right) = \text{gaussian}$$

$$\begin{cases} k_t - k_{ss} = 0 \\ k(0) = \delta_{s=0} \end{cases}$$



Indeed:

- $Y(t=0) = \int_{-\infty}^{\infty} Z(s) \underbrace{k(s,0)}_{\delta_{s=0}}\, ds = \langle \delta_{s=0}, Z(s) \rangle = Z(0) = Y_0$

- $\frac{dY}{dt} = \frac{d}{dt}\left[\int_{-\infty}^{\infty} Z(s)\, k(s,t)\, ds\right] = \int_{-\infty}^{\infty} Z(s) \frac{dk}{dt}(s,t)\, ds$
$= \int_{-\infty}^{\infty} Z(s)\, k_{ss}(s,t)\, ds$

$$= \int_{-\infty}^{\infty} Z_{ss}(s)\, k(s,t)\, ds$$

$$= -\int_{-\infty}^{\infty} AZ(s)\, k(s,t)\, ds$$

linearity and t-independence of A $\hookrightarrow$

$$= -A \int_{-\infty}^{\infty} Z(s)\, k(s,t)\, ds$$

$$= -AY(t)$$

This shows that

$$\frac{dY}{dt} + AY = 0, \quad t>0; \quad Y(0) = Y_0.$$

This transform can be also used in the control context to derive control properties of the heat equation out of those of the wave equation.

There is anyhow, a large literature on the control of heat-like equations, with specific parabolic methodologies, that do not need to use the corresponding theory for wave problems and transmutation formulas.