THE OPTIMAL CONTROL APPROACH

E. Zuazua, 19.07.2021

We have analyzed the controllability of

$$(1) \qquad x' + Ax = Bu$$

and we have seen that it can be characterized through the Kalman rank condition for (A, B).

We have also seen that the control of minimal $L^2(0,T;\mathbb{R}^m)$-norm driving the system from the initial configuration x_0 to the final one x_T, can be obtained through the minimization of the functional

$$(2) \quad J(\varphi_T) = \frac{1}{2}\int_0^T |B^*\varphi|^2 dt + \langle x_T, \varphi_T \rangle - \langle x_0, \varphi(0) \rangle$$

over all possible solutions of the adjoint system

$$(3) \qquad \begin{cases} -\varphi' + A^*\varphi = 0, & 0 < t < T \\ \varphi(T) = \varphi_T. \end{cases}$$

But this is often not the approach adopted in practical applications. There are several reasons for that at least:

- Checking the Kalman rank condition for the pair (A, B) can be algorithmically complex in high dimension.
- This is consistent with the fact that in infinite space dimensions things get very technical. The controllability theory varies significantly from parabolic to hyperbolic Partial Differential Equations (PDE), for instance.
- This theory is limited to linear problems although it can lead, of course, for local results in the nonlinear setting.
- Often those handling with applications are unaware of the adjoint methodology

It is therefore natural to think on a more straightforward procedure. It is in fact the one adopted in the context of Machine Learning where the problems are high-dimensional are nonlinear in nature.

Here we illustrate how this direct procedure would be in the context under consideration

$$(4) \qquad \begin{cases} x' + Ax = Bu, & 0 < t < T \\ x(0) = x_0, \end{cases}$$

the goal being to reach the target

$$(5) \qquad x(T) = x_T.$$

Given a constant $k > 0$ we could simply consider the functional

$$(6) \quad F_k(u) = \frac{1}{2}\int_0^T |u|^2 dt + \frac{k}{2}\| x(T) - x_T \|^2$$

and minimize it over $L^2(0, T; \mathbb{R}^m)$.

By doing so we would be finding a control $u \in L^2(0,T; \mathbb{R}^m)$ making a composite between not being too large and assuring that the target x_T is approximated by $x(T)$.

Of course the control $u = u_k$ and the solution $x = x_k$ will depend on the penalization constant $k > 0$.

In principle one expects that, as $k \nearrow$, the distance to the target will decrease: $\| x_k(T) - x_T \| \downarrow$.

Of course, by the DMCV, the minimizer of F_k exists and is unique. The minimizer could also be approximated by a gradient descent algorithm.

We observe however a big difference between this more straightforward approach and the duality one. The functional F_k needs to be minimized in the infinite-dimensional space $L^2(0,T;\mathbb{R}^m)$, while in the adjoint approach the functional J is minimized over $\mathbb{R}^n$, preserving the finite-dimensional nature of the problem.

When one adopts the direct optimization approach minimizing $F_k : L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$, we get u_k as control. But, how close is u_k from solving the controllability problem? In other words, can we estimate

$$\| x_k(T) - x_T \| ?$$

In practical applications one computes the minimum of F_k. But how does one choose k? How large should k be so to guarantee that $\| x_k(T) - x_T \|$ is small enough?

The following can be:

Theorem. If (A, B) is controllable, i.e. it fulfills the Kalman rank condition, then the minimizers u_k of F_k as $k \to \infty$ remain uniformly bounded in $L^2(0,T; \mathbb{R}^m)$ and

$$\| x_k(T) - x_T \| \leq \frac{C}{\sqrt{k}}.$$

In particular, the weak limit of u_k as $k \to \infty$ leads to an exact control assuring that $x(T) = x_T$.

Before we do the proof, which is actually quite simple, it is important to observe that when the system is not controllable, the minimizers u_k of F_k will be unbounded as $k \to \infty$ since the minimization process will try to force to diminish $\|x(T) - x_T\|$, which, in the absence of controllability, will never reach 0 if the target x_T is not reachable.

Proof. The system being controllable there is a control $u^*(t)$ (the one out of the adjoint methodology, for instance) such that

$$x^*(T) = x^T.$$

Obviously, by the definition of minimizer, we have

$$F_k(u_k) \leq F_k(u^*) = \frac{1}{2}\int_0^T |u^*|^2 dt = C.$$

Thus:

- $\int_0^T |u|^2 dt \leq C$

- $\frac{k}{2} \| x_k(T) - x_T \|^2 \leq C$

$$\Rightarrow \quad \| x_k(T) - x_T \| \leq \sqrt{\frac{2C}{k}}.$$