Finite-dimensional linear control

Enrique Zuazua

Chair of Dynamics, Control and Numerics
Department of Data Science
FAU
Erlangen, Germany
https://caa-avh.nat.fau.eu

Finite-dimensional control

Let $n, m \in \mathbb{N}^*$ and $T > 0$. Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0,T), \\ \qquad x(0) = x^0. \end{cases} \tag{1}$$

In (1), A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0,T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0,T] \longrightarrow \mathbb{R}^m$ the *control*. Both are vector functions of n and m components respectively depending exclusively on time t. Obviously, in practice $m \leq n$. The most desirable goal is, of course, controlling the system by means of a minimum number m of controls.

Given an initial datum $x^0 \in \mathbb{R}^n$ and a vector function $u \in L^2(0, T; \mathbb{R}^m)$, system (1) has a unique solution $x \in H^1(0, T; \mathbb{R}^n)$ characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0, T]. \tag{2}$$

System (1) is **exactly controllable** in time $T > 0$ if given any initial and final one $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0, T, \mathbb{R}^m)$ such that the solution of (1) satisfies $x(T) = x^1$.
According to this definition the aim of the control process consists in driving the solution x of (1) from the initial state x^0 to the final one x^1 in time T by acting on the system through the control u.

Example 1. Consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

Then the system

$$x' = Ax + Bu$$

can be written as

$$\begin{cases} x_1' = x_1 + u \\ x_2' = x_2, \end{cases}$$

or equivalently,

$$\begin{cases} x_1' = x_1 + u \\ x_2 = x_2^0 e^t, \end{cases}$$

where $x^0 = (x_1^0, x_2^0)$ are the initial data.
This system is not controllable since the control u does not act on the second component x_2 of the state which is completely determined by the initial data x_2^0.

Example 2. By the contrary, the equation of the harmonic oscillator is controllable

$$x'' + x = u. \tag{4}$$

The matrices A and B are now respectively

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One can easily check the controllability in this case by simply building a smooth curve $x = x(t)$ taking the initial values at $t = 0$ and the final ones at $t = T$, and then, computing a posteriori the control $u(t) = x''(t) + x(t)$.

In fact a scalar equation equation of arbitrary order

$$x^{(n)} + a_{n-1}x^{(n-1)} + ...a_1x' + a_0x = u(t).$$

Define the set of reachable states

$$R(T, x^0) = \{x(T) \in \mathbb{R}^n : x \text{ solution of (1) with } u \in (L^2(0, T))^m\}. \tag{5}$$

The exact controllability property is equivalent to the fact that $R(T, x^0) = \mathbb{R}^n$ for any $x^0 \in \mathbb{R}^n$.

Let A^* be the adjoint matrix of A, i.e. the matrix with the property that $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in \mathbb{R}^n$. Consider the following homogeneous adjoint system of (1):

$$\begin{cases} -\varphi' = A^*\varphi, \quad t \in (0, T) \\ \varphi(T) = \varphi_T. \end{cases} \tag{6}$$

This is an equivalent condition for exact controllability .

Lemma

An initial datum $x^0 \in \mathbb{R}^n$ of (1) is driven to zero in time T by using a control $u \in L^2(0, T)$ if and only if

$$\int_0^T \langle u, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall\varphi. \tag{7}$$

Proof:

Let φ_T be arbitrary in $\mathbb{R}^n$ and φ the corresponding solution of (6). By multiplying (1) by φ and (6) by x we deduce that

$$\langle x', \varphi \rangle = \langle Ax, \varphi \rangle + \langle Bu, \varphi \rangle; \quad -\langle x, \varphi' \rangle = \langle A^* \varphi, x \rangle.$$

Hence,

$$\frac{d}{dt} \langle x, \varphi \rangle = \langle Bu, \varphi \rangle$$

which, after integration in time, gives that

$$\langle x(T), \varphi_T \rangle - \langle x^0, \varphi(0) \rangle = \int_0^T \langle Bu, \varphi \rangle dt = \int_0^T \langle u, B^* \varphi \rangle dt. \quad (8)$$

We obtain that $x(T) = 0$ if and only if (7) is verified for any $\varphi_T \in \mathbb{R}^n$.

Identity (7) is in fact an optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$,

$$J(\varphi_T) = \frac{1}{2}\int_0^T \mid B^*\varphi \mid^2 dt + \langle x^0, \varphi(0)\rangle$$

where φ is the solution of the adjoint system (6) with initial data φ_T at time $t = T$.
More precisely:

Lemma

Suppose that J has a minimizer $\widehat{\varphi}_T \in \mathbb{R}^n$ and let $\widehat{\varphi}$ be the solution of the adjoint system (6) with initial data $\widehat{\varphi}_T$. Then

$$u = B^*\widehat{\varphi} \tag{9}$$

is a control of system (1) with initial data x^0.

Proof:

If $\widehat{\varphi}_T$ is a point where J achieves its minimum value, then

$$\lim_{h\to 0} \frac{J\left(\widehat{\varphi}_T + h\varphi_T\right) - J\left(\widehat{\varphi}_T\right)}{h} = 0, \quad \forall \varphi_T \in \mathbb{R}^n.$$

This is equivalent to

$$\int_0^T \langle B^*\widehat{\varphi}, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall \varphi_T \in \mathbb{R}^n,$$

which, in view of Lemma 1, implies that $u = B^*\widehat{\varphi}$ is a control for (1).

But minimizing the functional J requires of its coercivity.
System (6) is said to be **observable** in time $T > 0$ if there exists $c > 0$ such that

$$\int_0^T \mid B^*\varphi \mid^2 dt \geq c \mid \varphi(0) \mid^2, \tag{10}$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (6).
In the sequel (10) will be called the **observation** or **observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^*\varphi(t)$ for $0 < t < T$.

The following remark is very important in the context of finite dimensional control.
Inequality (10) is equivalent to the following unique continuation principle:

$$B^*\varphi(t) = 0, \ \forall t \in [0, T] \Rightarrow \varphi_T = 0. \tag{11}$$

This is an uniqueness or unique continuation property.
Unfortunately the equivalence is not true for infinite-dimensional systems (PDE, distributed parameter systems), as we use that $\partial B(0,1)$ is compact to prove that (11) implies (10).

UNIQUE CONTINUATION $\rightarrow$ OBSERVABILITY INEQUALITY $\rightarrow$ CONTROLLABILITY

WITH A CONSTRUCTIVE PROCEDURE TO BUILD CONTROLS BY MINIMIZING A COERCIVE FUNCTIONAL.

What about the observability property? Are there algebraic conditions on the state matrix A and the control one B for it to be true?

The following classical result is due to R. E. Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems.

Theorem

System (1) is exactly controllable in some time T if and only if

$$rank\,[B, AB, \cdots, A^{n-1}B] = n. \quad (12)$$

Consequently, if system (1) is controllable in some time $T > 0$ it is controllable in any time.

Proof of Theorem 3: " $\Leftarrow$ ".

Suppose now that $\text{rank}([B, AB, \cdots, A^{n-1}B]) = n$. It is sufficient to show that system (6) is observable.
Assume $B^*\varphi = 0$ and $\varphi(t) = e^{A^*(T-t)}\varphi_T$, it follows that $B^*e^{A^*(T-t)}\varphi_T \equiv 0$ for all $0 \leq t \leq T$. By computing the derivatives of this function in $t = T$ we obtain that

$$B^*[A^*]^k\varphi_T = 0 \quad \forall k \geq 0.$$

But since $\text{rank}\left(\left[B, AB, \cdots, A^{n-1}B\right]\right) = n$ we deduce that

$$\text{Ker}\left(\left[B^*, B^*A^*, \cdots, B^*(A^*)^{n-1}\right]\right) = \{0\}$$

and therefore $\varphi_T = 0$. Hence, (11) is verified.

Proof of Theorem 3: " $\Rightarrow$ "

Suppose that $\text{rank}([B, AB, \cdots, A^{n-1}B]) < n$. Then the kernel of the adjoint is non trivial. Accordingly, there is a non-trivial φ_T such that

$$B^*[A^*]^k\varphi_T = 0 \quad \forall k \geq 0.$$

Let φ be the corresponding solution of the adjoint system.
From Cayley-Hamilton Theorem we deduce that there exist constants $c_1, \cdots, c_n$ such that, $A^n = c_1A^{n-1} + \cdots + c_nI$. Expanding the exponential in power series it is the easy to see that for all $t \in (0, T)$:

$$B^*\varphi(t) = B^* \exp(A^*(T-t))\varphi_T \equiv 0.$$

Multiplying the state equation by φ it is easy to see that

$$< x(t), \varphi(t) >= x(t)^*\varphi(t) = (x^0)^*e^{A^*t}\varphi(t) + \int_0^t u(s)^*B^*e^{A^*(t-s)}\varphi(t)ds$$

$$= (x^0)^*e^{A^*t}\varphi(t) + \int_0^t u(s)^*B^*e^{A^*(T-s)}\varphi_T = (x^0)^*e^{A^*t}\varphi(t)$$

Consequently, the product $< x(t), \varphi(t) >$ is invariant of the control, which excludes the controllability property to hold. The proof of Theorem 3 is now complete.

The set of controllable pairs (A, B) is open and dense.

- Most systems are controllable;
- The controllability property is robust, i. e. it is invariant under small perturbations of A and/or B.

When controllability holds,

$$\| u \|_{L^2(0,T)} \leq C|e^{AT}x^0 - x^1| \tag{13}$$

for any initial data x^0 and final objective x^1.
Linear scalar equations of any order provide examples of systems that are controllable with only one control: k

$$x^{(k)} + a_1 x^{(k-1)} + \ldots + a_{k-1}x = u.$$

Exercise: Check that the Kalman condition is fulfilled in this case.

Bang-bang

Let us consider the particular case

$$B \in \mathcal{M}_{n\times 1}, \tag{14}$$

i. e. $m = 1$, in which only one control $u : [0, T] \to \mathbb{R}$ is available and B is a column vector.

To build bang-bang controls it is convenient to consider the quadratic functional:

$$J_{bb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \mid B^*\varphi \mid dt\right]^2 + \langle x^0, \varphi(0)\rangle \tag{15}$$

where φ is the solution of the adjoint system (6) with initial data φ_T.

The same argument as above shows that J_{bb} is also continuous and coercive. It follows that J_{bb} attains a minimum in some point $\widehat{\varphi}_T \in \mathbb{R}^n$.

The optimality condition (the Euler-Lagrange equations) its minimizers satisfy:

$$\int_0^T \mid B^*\widehat{\varphi} \mid dt \int_0^T \mathrm{sgn}\,(B^*\widehat{\varphi})B^*\psi(t)dt + \langle x^0, \varphi(0)\rangle = 0$$

for all $\varphi_T \in \mathbb{R}$, where φ is the solution of the adjoint system (6) with initial data φ_T. The control we are looking for is

$$u = \int_0^T \mid B^*\widehat{\varphi} \mid dt\,\mathrm{sgn}\,(B^*\widehat{\varphi})$$

where $\widehat{\varphi}$ is the solution of (6) with initial data $\widehat{\varphi}_T$.
The control is of bang-bang form, and takes only two values $\pm \int_0^T \mid B^*\widehat{\varphi} \mid dt$ switching finitely many times when the function $B^*\widehat{\varphi}$ changes sign. It has minimal $L^\infty(0, T)$ norm.

The control $u_2 = B^*\widehat{\varphi}$ obtained by minimizing the functional J has minimal $L^2(0,T)$ norm among all possible controls. Analogously, the control $u_\infty = \int_0^T \mid B^*\widehat{\varphi} \mid dt \, \mathrm{sgn}\,(B^*\widehat{\varphi})$ obtained by minimizing the functional J_{bb} has minimal $L^\infty(0,T)$ norm among all possible controls.

Proof: Let u be an arbitrary control for (1). Then (7) is verified both by u and u_2 for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J) in (7) we obtain that

$$\int_0^T < u, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u_2, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u, B^*\widehat{\varphi} > dt \leq ||u||_{L^2(0,T)}||B^*\widehat{\varphi}||_{L^2(0,T)}$$

$$= ||u||_{L^2(0,T)}||u_2||_{L^2(0,T)}$$

and the first part of the proof is complete.
For the second part a similar argument may be used. Indeed, let again u be an arbitrary control for (1). Then (7) is verified by u and u_∞ for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J_{bb}) in (7) we obtain that

$$\int_0^T B^*\widehat{\varphi} u dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_\infty||^2_{L^\infty(0,T)} = \left(\int_0^T |B^*\widehat{\varphi}| dt\right)^2 = \int_0^T B^*\widehat{\varphi} u_\infty dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$\begin{aligned} ||u_\infty||^2_{L^\infty(0,T)} &= \int_0^T B^*\widehat{\varphi}\, udt \\ &\leq ||u||_{L^\infty(0,T)} \int_0^T |B^*\widehat{\varphi}| dt = ||u||_{L^\infty(0,T)} ||u_\infty||_{L^\infty(0,T)}, \end{aligned}$$

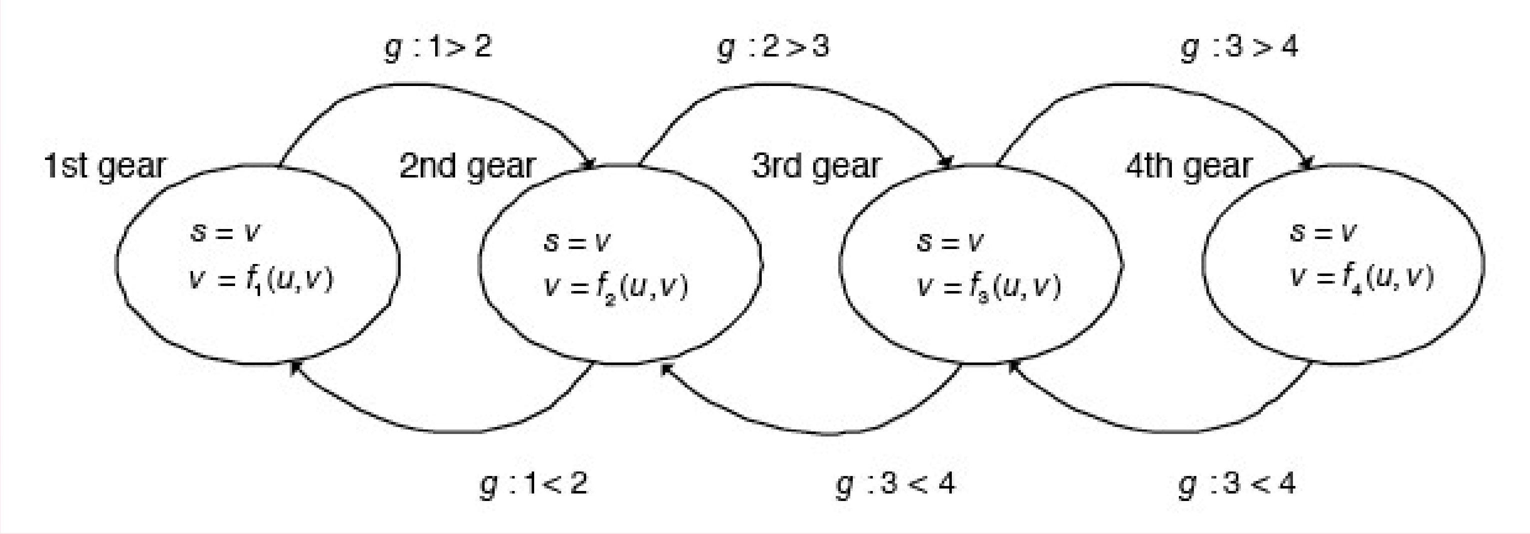
and the proof finishes.

Switching control

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = Ax(t) + u_1(t)b_1 + u_2(t)b_2 \\ x(0) = x^0. \end{cases} \tag{16}$$

$x(t) = (x_1(t), \ldots, x_N(t)) \in \mathbb{R}^N$ is the state of the system, A is a $N \times N-$matrix, $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are two scalar controls an b_1, b_2 are given control vectors in $\mathbb{R}^N$.
More general and complex systems may also involve switching in the state equation itself:



$$x'(t) = A(t)x(t) + u_1(t)b_1 + u_2(t)b_2, \quad A(t) \in \{A_1, ..., A_M\}.$$

Controllability

Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for control pairs (u_1, u_2) such that the solution of (16) satisfies

$$x(T) = x^1. \tag{17}$$

In the absence of constraints, controllability holds if and only if the Kalman rank condition is satisfied

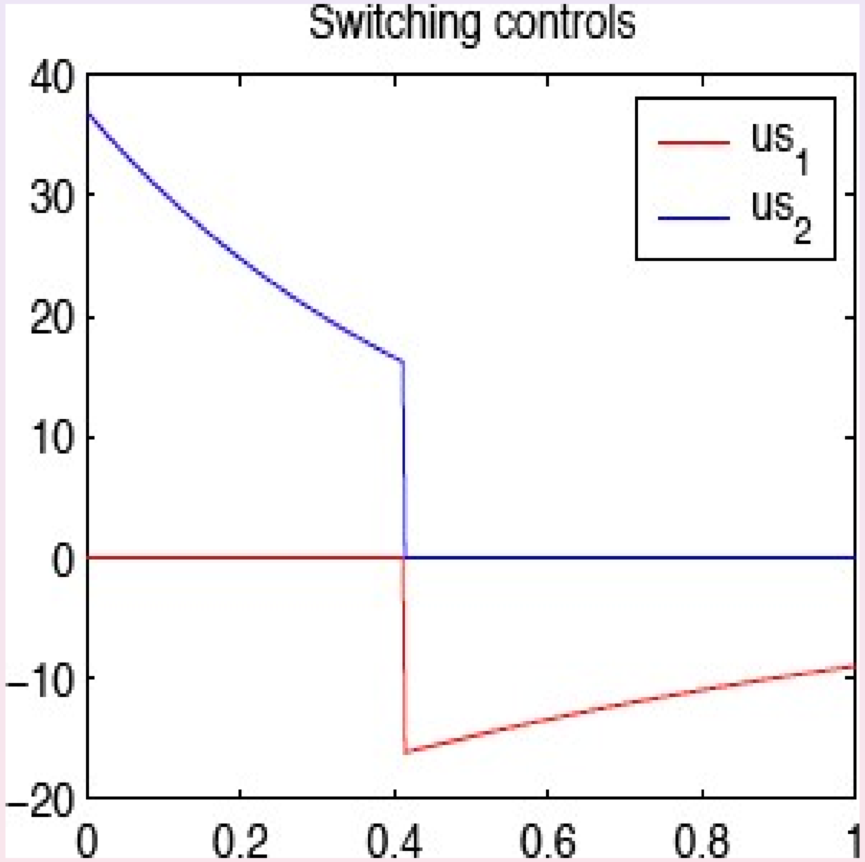
$$rank\left[B, AB, \ldots, A^{N-1}B\right] = N \tag{18}$$

with $B = (b_1, b_2)$.
We look for **switching controls:**

$$u_1(t)u_2(t) = 0, \qquad \text{a.e.} \quad t \in (0, T). \tag{19}$$

Under the rank condition above, these switching controls always exist.
To develop systematic strategies allowing to build switching controllers.

The controllers of a system endowed with different actuators are said to be of switching form when only one of them is active in each instant of time.



The classical theory guarantees that the standard controls $(u_1,\ u_2)$ may be built by minimizing the functional

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T \left[|b_1\cdot\varphi(t)|^2 + |b_2\cdot\varphi(t)|^2\right] dt - x^1\cdot\varphi^0 + x^0\cdot\varphi(0),$$

among the solutions of the adjoint system

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), & t\in(0,\ T) \\ \varphi(T) = \varphi^0. \end{cases} \tag{20}$$

The rank condition for the pair (A, B) is equivalent to the following unique continuation property for the adjoint system which suffices to show the coercivity of the functional:

$$b_1\cdot\varphi(t) = b_2\cdot\varphi(t) = 0, \quad \forall t\in[0,\ T] \to \varphi\equiv 0.$$

Preassigned switching

Given a partition $\tau = \{t_0 = 0 < t_1 < t_2 < ... < t_{2N} = T\}$ of the time interval $(0, T)$, consider the functional

$$J_\tau\left(\varphi^0\right) = \frac{1}{2}\sum_{j=0}^{N-1}\int_{t_{2j}}^{t_{2j+1}} \left|b_1 \cdot \varphi(t)\right|^2 dt + \frac{1}{2}\sum_{j=0}^{N-1}\int_{t_{2j+1}}^{t_{2j+2}} \left|b_2 \cdot \varphi(t)\right|^2 dt$$

$$-x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0).$$

Under the same rank condition this functional is coercive too. In fact, in view of the time-analiticity of solutions, the above unique continuation property implies the apparently stronger one:

$$b_1 \cdot \varphi(t) = 0 \quad t \in (t_{2j}, t_{2j+1});\ b_2 \cdot \varphi(t) = 0 \quad t \in (t_{2j+1}, t_{2j+2}) \rightarrow \varphi \equiv 0$$

and this one suffices to show the coercivity of J_τ. Thus, J_τ has an unique minimizer $\check{\varphi}$ and this yields the controls

$$u_1(t) = b_1 \cdot \check{\varphi}(t),\ t \in (t_{2j}, t_{2j+1}); \quad u_2(t) = b_2 \cdot \check{\varphi}(t),\ t \in (t_{2j+1}, t_{2j+2})$$

which are obviously of switching form.

A new functional for automatic switching

Consider now, without an a priori partition of $[0, T]$:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\left(\left|b_1 \cdot \varphi(t)\right|^2, \left|b_2 \cdot \varphi(t)\right|^2\right) dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0). \tag{21}$$

Theorem

Assume that the pairs $(A, b_2 - b_1)$ and $(A, b_2 + b_1)$ satisfy the rank condition. Then, for all $T > 0$, J_s achieves its minimum. Furthermore, the switching controllers

$$\begin{cases} u_1(t) = \tilde{\varphi}(t) \cdot b_1 & \text{when} \quad \left|\tilde{\varphi}(t) \cdot b_1\right| > \left|\tilde{\varphi}(t) \cdot b_2\right| \\ u_2(t) = \tilde{\varphi}(t) \cdot b_2 & \text{when} \quad \left|\tilde{\varphi}(t) \cdot b_2\right| > \left|\tilde{\varphi}(t) \cdot b_1\right| \end{cases} \tag{22}$$

where $\tilde{\varphi}$ is the solution of (20) with datum $\tilde{\varphi}^0$ at time $t = T$, control the system.

1. The rank condition on the pairs $\left(A,\ b_2 \pm b_1\right)$ is a necessary and sufficient condition for the controllability of the systems

$$x' + Ax = \left(b_2 \pm b_1\right) u(t). \tag{23}$$

   This implies that the system with controllers b_1 and b_2 is controllable too but the reverse is not true (take $b_2 = \pm b_1$).

2. The rank conditions on the pairs $\left(A,\ b_2 \pm b_1\right)$ are needed to ensure that the set

$$\left\{ t \in (0,\ T) : \left|\varphi(t) \cdot b_1\right| = \left|\varphi(t) \cdot b_2\right| \right\} \tag{24}$$

   is of null measure, which ensures that the controls in (22) are genuinely of switching form.

Sketch of the proof

There are two key points:

a) Showing that the functional J_s is coercive, i. e.,

$$\lim_{\|\varphi^0\|\to\infty} \frac{J_s(\varphi^0)}{\| \varphi^0 \|} = \infty,$$

which guarantees the existence of minimizers.

Coercivity is immediate since

$$|\varphi(t)\cdot b_1|^2 + |\varphi(t)\cdot b_2|^2 \leq 2\max\left[|\varphi(t)\cdot b_1|^2, \, |\varphi(t)\cdot b_2|^2\right]$$

and, consequently, the functional J_s is bounded below by a functional equivalent to the classical one J.

b) Showing that the controls obtained by minimization are of switching form.

This is equivalent to proving that the set

$$I = \{t \in (0,\ T) : |\tilde{\varphi} \cdot b_1| = |\tilde{\varphi} \cdot b_2|\}$$

is of null measure.

Assume for instance that the set
$I_+ = \{t \in (0,\ T) : \tilde{\varphi}(t) \cdot (b_1 - b_2) = 0\}$ is of positive measure, $\tilde{\varphi}$ being the minimizer of J_s. The time analyticity of $\tilde{\varphi} \cdot (b_1 - b_2)$ implies that $I_+ = (0,\ T)$. Accordingly $\tilde{\varphi} \cdot (b_1 - b_2) \equiv 0$ and, consequently, taking into account that the pair $(A,\ b_1 - b_2)$ satisfies the Kalman rank condition, this implies that $\tilde{\varphi} \equiv 0$. This would imply that

$$J(\varphi^0) \geq 0, \forall \varphi^0 \in \mathbb{R}^N$$

which may only happen in the trivial situation in which $x^1 = e^{AT} x^0$, a trivial situation that we may exclude.

The Euler-Lagrange equations associated to the minimization of J_s take the form

$$\int_{S_1} \tilde{\varphi}(t)\cdot b_1\, \psi(t)\cdot b_1\, dt + \int_{S_2} \tilde{\varphi}(t)\cdot b_2\, \psi(t)\cdot b_2\, dt - x^1\cdot\psi^0 + x^0\cdot\psi(0) = 0,$$

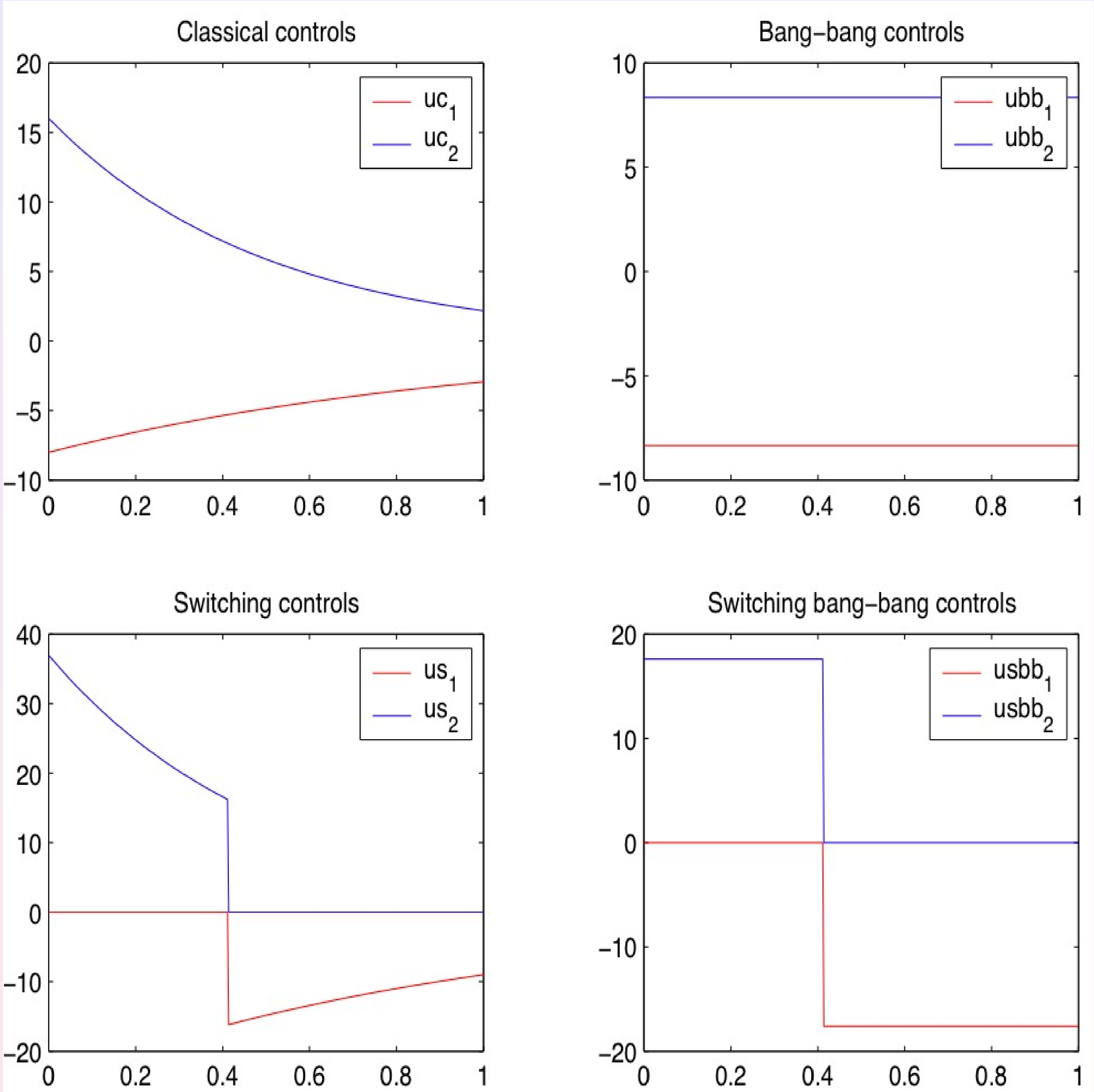
for all $\psi^0 \in \mathbb{R}^N$, where

$$\begin{cases} S_1 = \{t \in (0,\ T) :\ |\tilde{\varphi}(t)\cdot b_1| > |\tilde{\varphi}(t)\cdot b_2|\}, \\ S_2 = \{t \in (0,\ T) :\ |\tilde{\varphi}(t)\cdot b_1| < |\tilde{\varphi}(t)\cdot b_2|\}. \end{cases} \tag{25}$$

In view of this we conclude that

$$u_1(t) = \tilde{\varphi}(t)\cdot b_1\, 1_{S_1}(t), \quad u_2(t) = \tilde{\varphi}(t)\cdot b_2\, 1_{S_2}(t), \tag{26}$$

where 1_{S_1} and 1_{S_2} stand for the characteristic functions of the sets S_1 and S_2, are such that the switching condition holds and the corresponding solution satisfies the final control requirement.


Classical controls
uc1
uc2
Bang–bang controls
ubb1
ubb2
Switching controls
us1
us2
Switching bang–bang controls
usbb1
usbb2

Slight variants of these arguments lead to switching controls of different nature, in particular to switching bang-bang controls. For instance, when minimizing the functional

$$J_{sb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \max\left(|\varphi(t)\cdot b_1|,\, |\varphi(t)\cdot b_2|\right) dt\right]^2 - x^1\cdot\varphi^0 + x^0\cdot\varphi(0),$$

the controls take the form

$$u_1(t) = \lambda\,\mathrm{sgn}\left(\tilde\varphi(t)\cdot b_1\right) 1_{S_1}(t);\; u_2(t) = \lambda\,\mathrm{sgn}\left(\tilde\varphi(t)\cdot b_2\right) 1_{S_2}(t).$$

where

$$\lambda = \int_0^T \max\left(|\tilde\varphi(t)\cdot b_1|,\, |\tilde\varphi(t)\cdot b_2|\right) dt.$$

Optimality

The switching controls we obtain this way are of minimal $L^2\left(0,\ T;\ \mathbb{R}^2\right)$-norm, the space $\mathbb{R}^2$ being endowed with the ℓ^1 norm, i. e. with respect to the norm

$$||(u_1, u_2)||_{L^2(0,\ T;\ \ell^1)} = \left[\int_0^T (|\tilde{u}_1| + |\tilde{u}_2|)^2 dt\right]^{1/2}.$$

Switching bang-bang controls are of minimal $L^\infty\left(0,\ T;\ \mathbb{R}^2\right)$-norm

Stabilisation

The controls we have obtained so far are the so called open loop controls. In practice, it is interesting to get closed loop or feedback controls, so that its value is realted in real time with the state itself.

In this section we assume that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$. Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{27}$$

When $u \equiv 0$, the energy of the solution of (27) is conserved. Indeed, by multiplying (27) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt}|x(t)|^2 = 0. \tag{28}$$

Hence,

$$|x(t)| = |x^0|, \quad \forall t \geq 0. \tag{29}$$

The problem of *stabilization* can be formulated in the following way. Suppose that the pair (A, B) is controllable. We then look for a matrix L such that the solution of system (27) with the *feedback* control law

$$u(t) = Lx(t) \tag{30}$$

has **a uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x^0| \tag{31}$$

for any solution.

Note that, according to the law (30), the control u is obtained in real time from the state x.

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx \tag{32}$$

has an uniform exponential decay rate.
Remark that we cannot expect more than (31). Indeed, for instance, the solutions of (32) may not satisfy $x(T) = 0$ in finite time T. Indeed, if it were the case, from the uniqueness of solutions of (32) with final state 0 in $t = T$, it would follow that $x^0 \equiv 0$.

Theorem

If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^*$ stabilizes the system, i.e. the solution of

$$\begin{cases} x' = Ax - BB^*x \\ x(0) = x^0 \end{cases} \tag{33}$$

has an uniform exponential decay (31).

Proof

With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = - < BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

Hence, the norm of the solution decreases in time.
Moreover,

$$|x(T)|^2 - |x(0)|^2 = -2\int_0^T \mid B^*x \mid^2 dt. \tag{34}$$

To prove the uniform exponential decay it is sufficient to show that there exist $T > 0$ and $c > 0$ such that

$$|x(0)|^2 \leq c\int_0^T \mid B^*x \mid^2 dt \tag{35}$$

for any solution x of (33). Indeed, from (34) and (35) we would obtain that

$$|x(T)|^2 - |x(0)|^2 \leq -\frac{2}{c}|x(0)|^2 \tag{36}$$

and consequently

$$|x(T)|^2 \leq \gamma |x(0)|^2 \tag{37}$$

with

$$\gamma = 1 - \frac{2}{c} < 1. \tag{38}$$

Hence,

$$|x(kT)|^2 \leq \gamma^k |x^0|^2 = e^{(\ln \gamma)k} |x^0|^2 \quad \forall k \in \mathbb{N}. \tag{39}$$

Now, given any $t > 0$ we write it in the form $t = kT + \delta$, with $\delta \in [0, T)$ and $k \in \mathbb{N}$ and we obtain that

$$|x(t)|^2 \leq |x(kT)|^2 \leq e^{-|\ln(\gamma)|k} |x^0|^2 =$$

$$= e^{-|\ln(\gamma)|\left(\frac{t}{T}\right)} e^{|\ln(\gamma)|\frac{\delta}{T}} |x^0|^2 \leq \frac{1}{\gamma} e^{-\frac{|\ln(\gamma)|}{T} t} |x^0|^2.$$

We have obtained the desired decay result (31) with

$$c = \frac{1}{\gamma}, \omega = \frac{|\ln(\gamma)|}{T}. \tag{40}$$

To prove (35) we decompose the solution x of (33) as $x = \varphi + y$ with φ and y solutions of the following systems:

$$\begin{cases} \varphi' = A\varphi \\ \varphi(0) = x^0, \end{cases} \tag{41}$$

and

$$\begin{cases} y' = Ay - BB^*x \\ y(0) = 0. \end{cases} \quad (42)$$

Remark that, since A is skew-adjoint, (41) is exactly the adjoint system (6) except for the fact that the initial data are taken at $t = 0$.

As we have seen in the proof of Theorem 3, the pair (A, B) being controllable, the following observability inequality holds for system (41):

$$|x^0|^2 \leq C \int_0^T |\, B^*\varphi \,|^2 \, dt. \quad (43)$$

Since $\varphi = x - y$ we deduce that

$$|x^0|^2 \leq 2C \left[\int_0^T |\, B^*x \,|^2 \, dt + \int_0^T |\, B^*y \,|^2 \, dt \right].$$

On the other hand, it is easy to show that the solution y of (42) satisfies:

$$\frac{1}{2}\frac{d}{dt} \mid y \mid^2 = -\langle B^*x, B^*y\rangle \leq |B^*x|\,|B^*|\,|y| \leq \frac{1}{2}\left(|y|^2 + |B^*|^2|B^*x|^2\right).$$

From Gronwall's inequality we deduce that

$$\mid y(t) \mid^2 \leq |B^*|^2 \int_0^t e^{t-s} \mid B^*x \mid^2 ds \leq |B^*|^2 e^T \int_0^T \mid B^*x \mid^2 dt \qquad (44)$$

and consequently

$$\int_0^T \mid B^*y \mid^2 dt \leq |B|^2 \int_0^T \mid y \mid^2 dt \leq T|B|^4 e^T \int_0^T \mid B^*x \mid^2 dt.$$

Finally, we obtain that

$$\mid x^0 \mid^2 \leq 2C \int_0^T \mid B^*x \mid^2 dt + C|B^*|^4 e^T T \int_0^T \mid B^*x \mid^2 dt \leq C' \int_0^T \mid B^*x \mid^2 dt$$

and the proof of Theorem 5 is complete.

Example

Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{45}$$

where m, k and R are positive constants.
Note that (45) may be written in the equivalent form

$$mx'' + Rx = -kx'$$

which indicates that an applied force, proportional to the velocity of the point-mass and of opposite sign, is acting on the oscillator.

It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_{\pm} = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\mathrm{Re}\, r_{\pm} = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \leq 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \geq 4mR. \end{cases}$$

We observe here the classical overdamping phenomenon. Contradicting a first intuition it is not true that the decay rate increases when the value of the damping parameter k increases.

Arbitrary decay rate

If (A, B) is controllable, we have proved the uniform stability property of the system (27), under the hypothesis that A is skew-adjoint. However, this property holds even if A is an arbitrary matrix. More precisely, we have:

Theorem

If (A, B) is controllable then it is also stabilizable. Moreover, it is possible to prescribe any complex numbers λ_1, λ_2,...,λ_n as the eigenvalues of the closed loop matrix $A + BL$ by an appropriate choice of the feedback matrix L so that the decay rate may be made arbitrarily fast.

This result is not in contradiction with the behavior we observed above on the harmonic oscillator (the overdamping phenomenon). In order to obtain the arbitrarily fast decay one needs to use all components of the state on the feedback law!

Conclusions

We have shown that:

- Controllability and observabilitiy are equivalent notions (Wiener's cybernetics).
- Both hold for all T if and only if the Kalman rank condition is fulfilled.
- The controls may be obtained as minimizers of suitable quadratic functionals over the space of solutions of the adjoint system.
- There are very many controls: smooth ones, in bang-bang form,...
- When the system is endowed with various actuators one may establish automatic strategies to switch from one to another.
- Controllable systems are stabillizable by means of closed loop or feedback controls.

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